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Global existence of a class of smooth 3D spherically symmetric flows of Chaplygin gases with variable entropy

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Abstract

We consider smooth three-dimensional spherically symmetric flows of Chaplygin gases with variable entropy, whose initial data are obtained by adding a small smooth perturbation with compact support to a constant state. For ideal polytropic gases, the lifespan would generally be finite, but, for Chaplygin gases, we show that the lifespan is infinite. This is done by constructing and estimating a suitable approximate flow. The special form of the nonlinearities yields "null conditions" thanks to which some decay estimates valid in the ideal polytropic case can be improved. This enables us to prove global existence.

Résumé

Nous considérons des écoulements tridimensionnels lisses, à symétrie sphérique, de gaz de Chaplygin à entropie variable, dont les données initiales s'obtiennent en ajoutant une petite perturbation lisse à support compact à un état constant. Pour des gaz idéaux polytropiques, la durée de vie serait en général finie, mais, pour les gaz de Chaplygin, nous montrons que la durée de vie est infinie. Ceci se fait en construisant et en estimant un écoulement approché convenable. La forme spéciale des non-linéarités crée des « conditions nulles » grâce auxquelles certaines estimations de décroissance, valables dans le cas idéal polytropique, peuvent être améliorées. Ceci nous permet de démontrer l'existence globale.

MSC: 76N10; 35Q35; 35L99

Keywords: Chaplygin gases with variable entropy; Global spherically symmetric solutions; Null conditions; Decay estimates

1. Introduction

Smooth Eulerian flows of ideal polytropic gases with initial data close to a constant state do not, in general, exist for all positive times (cf. e.g. [14,1,15,5] and references given there). In [5], an asymptotic formula for the lifespan was obtained (for suitable initial data close to a constant state) in the 3D spherically symmetric case with variable entropy. When the adiabatic constant γ of the gas is equal to -1 (and the equation of state is suitably modified to keep the pressure and its derivative with respect to the density positive), the gas in a so-called Chaplygin gas, also

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called Karman–Tsien gas (see [13,12], and in the isentropic case, also [3]). The purpose of this paper is to show that there is actually global existence in the case of 3D Chaplygin gases, provided that the initial data are a small smooth spherically symmetric perturbation with compact support of a constant state. Thus we extend to the 3D spherically symmetric case a well-known 1D result; in 1D, it is an immediate consequence of the results of [10], since all the eigenvalues of the coefficient of ∂_x in the 1D version of (2.1)–(2.3) below are linearly degenerate in the case of Chaplygin gases.

Notice that it has been proved in [6] that suitable initial data, which force particles to spread out, lead to global existence in the future for ideal polytropic gases; the assumptions in [6] are completely different from those of our paper.

To prove our result, we shall use a suitable approximate solution (similar to the one introduced in [5]). It will turn out that the condition $\gamma = -1$ is just a "null condition" on nonlinearities; actually in the region where the flow is isentropic, we can introduce a potential and the nonlinearities of the potential equation satisfy a null condition in the sense of [7,2]. This will enable us to obtain improved decay estimates which will play a key role in the proof of our global existence result. We shall rely heavily on estimates from [9,16,17].

Our paper is organized as follows. In Section 2, we state our result precisely. In Section 3, we introduce the approximate solution. The various estimates leading to our result are proved in Sections 4–6. In order to avoid unnecessary interruptions, we have collected some useful auxiliary results in Appendices A and B at the end of the paper.

2. Statement of the result

We consider the compressible Euler equations:

$$\partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0, \tag{2.1}$$

$$\partial_t u + u \cdot \nabla u + \frac{1}{\rho} \nabla P = 0, \tag{2.2}$$

$$\partial_t S + u \cdot \nabla S = 0, \tag{2.3}$$

where $\rho > 0$ is the density, u the velocity, S the entropy, P the pressure; they are all functions of (x, t). We assume that $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geqslant 0$, and write $\nabla = (\partial_1, \partial_2, \partial_3)$. We consider Chaplygin gases (also called Karman–Tsien gases), that is we assume that $P = P_0 - \frac{A(S)}{\rho}$, where P_0 is a strictly positive constant, $A \in C^\infty(\mathbb{R})$ and A is everywhere > 0 (cf. [13,12], and, in the isentropic case, also [3]). In [5] we considered the ideal polytropic case, namely we assumed that $P = A\rho^\gamma e^{\mu S}$ where A, γ, μ are strictly positive constants and $\gamma > 1$. The present situation corresponds to the case $\gamma = -1$ (with a modification designed to keep P and $\frac{\partial P}{\partial \rho} > 0$). Fix $\bar{S} \in \mathbb{R}$ and assume that $A'(\bar{S}) \neq 0$. Also fix $M > 0$, $\bar{\rho} > 0$ such that $P_0 - \frac{A(\bar{S})}{\bar{\rho}} > 0$, and let $\varepsilon > 0$ be a small parameter, always assumed to belong to $(0, \bar{\varepsilon}]$ for some small $\bar{\varepsilon} \in (0, 1)$ throughout this paper. We shall consider initial conditions of the form:

$$\rho(x, 0) = \bar{\rho} + \varepsilon \rho_0(x, \varepsilon), \tag{2.4}$$

$$u(x, 0) = \varepsilon u_0(x, \varepsilon), \tag{2.5}$$

$$S(x, 0) = \bar{S} + \varepsilon S_0(x, \varepsilon), \tag{2.6}$$

where $\rho(x, 0) > 0$, $\rho_0(x, \varepsilon) = \rho^0(r) + \varepsilon \rho^1(r, \varepsilon)$, $u_0(x, \varepsilon) = (u^0(r) + \varepsilon u^1(r, \varepsilon))\frac{x}{r}$, $S_0(x, \varepsilon) = S^0(r) + \varepsilon S^1(r, \varepsilon)$; here ρ^j, $u^j \frac{x}{r}$, S^j, which are fixed throughout this paper, are C^∞ functions of x and vanish when $|x| \geqslant M$, and ρ^0, u^0, S^0 are independent of ε. We shall assume that for some $\bar{\varepsilon} > 0$, $C_\alpha > 0$, $|\partial_x^\alpha \rho^1| + |\partial_x^\alpha (u^1 \frac{x}{r})| + |\partial_x^\alpha S^1| \leqslant C_\alpha$ if $x \in \mathbb{R}^3$, $\alpha \in \mathbb{N}^3$ and $\varepsilon \leqslant \bar{\varepsilon}$.

The purpose of this paper is to prove the following result:

Theorem 2.1. *If $\varepsilon > 0$ is small.* (2.1)–(2.6) *has a* (*unique*) $C^\infty(\mathbb{R}^3 \times [0, +\infty))$ *solution.*

Actually, if $T_\varepsilon = \sup\{T > 0,\ (2.1)\text{–}(2.6) \text{ has a } C^\infty(\mathbb{R}^3 \times [0, T]) \text{ solution}\}$, one can obtain the much weaker result $\lim_{\varepsilon \xrightarrow{>} 0} \varepsilon \ln T_\varepsilon = +\infty$ by studying the problem (3.11)–(3.16) below as in [5]. In fact Theorem 2.1 will be obtained by improving estimates from [5].

Remark 2.1. If (ρ, u, S) is a smooth solution to (2.1)–(2.6) in $\mathbb{R}^3 \times [0, T]$, ρ, S are radial and u is of the form $u(x,t) = U(r,t)\frac{x}{r}$ (cf. Remark 2 of [5]).

Remark 2.2. Theorem 2.1 still holds if (2.5) is replaced by the initial condition $u(x,0) = \bar{u} + \varepsilon u_0(x,\varepsilon)$ where $\bar{u} \in \mathbb{R}^3$. Indeed, as is well known, replacing x by $x - t\bar{u}$ and u by $u - \bar{u}$ leads to a new solution of (2.1)–(2.3) (with the same pressure law), and so we immediately reduce to the assumptions of Theorem 2.1.

3. The approximate solution

Put $\bar{c} = (\frac{\partial P}{\partial \rho}(\bar{\rho}, \bar{S}))^{1/2}$, that is $\bar{c} = \frac{(A(\bar{S}))^{1/2}}{\bar{\rho}}$. Consider the vector fields $X = t\partial_t + \sum_{1\leqslant j\leqslant 3} x_j \partial_j$, $\Omega_1 = x_2\partial_3 - x_3\partial_2$, $\Omega_2 = x_3\partial_1 - x_1\partial_3$, $\Omega_3 = x_1\partial_2 - x_2\partial_1$, $\bar{L}_j = \bar{c}t\partial_j + \frac{x_j}{\bar{c}}\partial_t$ if $1 \leqslant j \leqslant 3$, and define $(\bar{\Lambda}_1, \dots, \bar{\Lambda}_{11}) = (\partial_t, \partial_1, \partial_2, \partial_3, X, \Omega_1, \Omega_2, \Omega_3, \bar{L}_1, \bar{L}_2, \bar{L}_3)$. If $a = (a_1, \dots, a_{11}) \in \mathbb{N}^{11}$, we set $\bar{\Lambda}^a = \bar{\Lambda}_1^{a_1} \dots \bar{\Lambda}_{11}^{a_{11}}$. If $y \in \mathbb{R}^N$, we shall denote by $|y|$ its Euclidean norm. We shall also write $|\cdot|$ for the corresponding operator norm for matrices. For a function $f(x,t)$, we shall set $|f(t)| = \sup_{x\in\mathbb{R}^3} |f(x,t)|$, $\|f(t)\| = (\int_{\mathbb{R}^3} |f(x,t)|^2\,\mathrm{d}x)^{1/2}$. For functions of x only, $\|\cdot\|$ will be the standard $L^2(\mathbb{R}^3_x)$ norm. Finally we shall write $\langle\xi\rangle = 1 + |\xi|$ if $\xi \in \mathbb{R}$ and $\partial = (\partial_t, \partial_1, \partial_2, \partial_3)$ (sometimes, when convenient, we shall also write ∂_0 instead of ∂_t).

As in [5], we denote by $(\rho_1, u_1, \bar{S})$ the solution (with constant entropy $\bar{S}$ and pressure P_1) of (2.1)–(2.3), such that $u_1 = u$ and $P_1 = P$ when $t = 0$. It follows that

$$\rho_1(x,0) = \big(\bar{\rho} + \varepsilon\rho_0(x,\varepsilon)\big)\frac{A(\bar{S})}{A(\bar{S} + \varepsilon S_0(x,\varepsilon))}, \tag{3.1}$$

$$u_1(x,0) = \varepsilon u_0(x,\varepsilon). \tag{3.2}$$

We have the following result:

Proposition 3.1. *One can find $\varepsilon_0, C_a, C_{k\delta} > 0$ such that the following holds: if $\varepsilon \leqslant \varepsilon_0$, ρ_1, u_1 exist for all $(x,t) \in \mathbb{R}^3 \times [0, +\infty)$ and satisfy the estimates:*

(1) $(|\bar{\Lambda}^a(\rho_1 - \bar{\rho})| + |\bar{\Lambda}^a u_1|)(x,t) \leqslant C_a\varepsilon\langle\bar{c}t - |x|\rangle^{-2}\langle\bar{c}t + |x|\rangle^{-1}$,
(2) $(|\partial\bar{\Lambda}^a(\rho_1 - \bar{\rho})| + |\partial\bar{\Lambda}^a u_1|)(x,t) \leqslant C_a\varepsilon\langle t\rangle^{-4}$ *if* $|x| \leqslant \frac{\bar{c}t}{2} + M + 1$,
(3) $|X^k u_1(x,t)| \leqslant C_{k\delta}\varepsilon\langle t\rangle^{-4+\delta}$ *if* $\delta < 1$ *and* $|x| \leqslant M\langle\bar{c}t\rangle^\delta$.

Proof. Denote by V_1 the potential function vanishing for large x and defined by the relations $\nabla V_1 = u_1$, $\partial_t V_1 = -\frac{1}{2}|u_1|^2 - h(\rho_1)$, where

$$h(\rho) = \int_{\bar{\rho}}^{\rho} \frac{1}{\sigma}\frac{\partial P}{\partial \sigma}\big(\sigma, \bar{S}\big)\,\mathrm{d}\sigma = \frac{\bar{c}^2}{2} - \frac{A(\bar{S})}{2\rho^2}.$$

From (2.1) it follows that

$$\partial_t^2 V_1 + 2\sum_{1\leqslant j\leqslant 3} \partial_j V_1\, \partial_t\partial_j V_1 + \sum_{1\leqslant j,k\leqslant 3} \partial_j V_1 \partial_k V_1 \partial_{jk}^2 V_1 - \big(\bar{c}^2 + 2\partial_t V_1 + |\nabla V_1|^2\big)\Delta V_1 = 0, \tag{3.3}$$

and of course, because of (3.2), (3.1):

$$V_1(x,0) = -\varepsilon\int_{|x|}^{M} \big(u^0(\lambda) + \varepsilon u^1(\lambda,\varepsilon)\big)\,\mathrm{d}\lambda, \tag{3.4}$$

$$\partial_t V_1(x,0) = -h\bigg(\big(\bar{\rho} + \varepsilon\rho_0(x,\varepsilon)\big)\frac{A(\bar{S})}{A(\bar{S} + \varepsilon S_0(x,\varepsilon))}\bigg) - \frac{1}{2}\varepsilon^2\big|u_0(x,\varepsilon)\big|^2. \tag{3.5}$$

Now the operator in (3.3) satisfies the null condition [7,2] and Proposition 3.1(1) follows at once from Theorem A.1 of Appendix A. (2) follows from (1) if we use the estimate (from [8]) $|\partial\varphi(x,t)| \leqslant \frac{C}{\langle\bar{c}t - |x|\rangle}\sum_{|b|=1} |\bar{\Lambda}^b\varphi(x,t)|$. Finally, writing $u_1(x,t) = U_1(r,t)\frac{x}{r}$, we have $X^k u_1(x,t) = (\int_0^r (\partial_\lambda X^k U_1)(\lambda,t)\,d\lambda)\frac{x}{r}$, so (3) follows from (2) for t large and from (1) for t small. $\square$

It is convenient to transform (2.1)–(2.3) (cf. [5] for a similar procedure in the ideal polytropic case). Set $\theta(x,t) = 1 - \frac{A(S)\bar\rho}{A(\bar S)\rho}(x, \frac{t}{c})$, $w(x,t) = \frac{1}{c}u(x,\frac{t}{c})$, $z(x,t) = \frac{A(\bar S)}{A(S)}(x,\frac{t}{c}) - 1$. It follows from (2.1)–(2.3) that

$$\partial_t\theta + w\cdot\nabla\theta + (1-\theta)\nabla\cdot w = 0, \tag{3.6}$$

$$\partial_t w + w\cdot\nabla w + (1-\theta)(1+z)\nabla\theta = 0, \tag{3.7}$$

$$\partial_t z + w\cdot\nabla z = 0. \tag{3.8}$$

Division of (3.7) by $1+z$ later will make (3.6)–(3.8) symmetric hyperbolic with respect to t. (2.4)–(2.6) give:

$$\theta(x,0) = \varepsilon\theta_0(x,\varepsilon), \tag{3.9}$$

$$w(x,0) = \varepsilon w_0(x,\varepsilon), \tag{3.10}$$

$$z(x,0) = \varepsilon z_0(x,\varepsilon), \tag{3.11}$$

where $\theta_0 = \frac{1}{\varepsilon}(1 - \frac{A(\bar S+\varepsilon S_0)\bar\rho}{A(\bar S)(\bar\rho+\varepsilon\rho_0)})$, $w_0 = \frac{1}{c}u_0$, $z_0 = \frac{1}{\varepsilon}(\frac{A(\bar S)}{A(\bar S+\varepsilon S_0)} - 1)$.

Let $(\theta_1, w_1, 0)$ correspond to $(\rho_1, u_1, \bar S)$, so $\theta_1(x,t) = 1 - \frac{\bar\rho}{\rho_1}(x,\frac{t}{c})$, $w_1(x,t) = \frac{1}{c}u_1(x,\frac{t}{c})$. Set $\sigma_\pm(x,t) = \langle t \pm |x|\rangle$, $\Lambda_k = \bar\Lambda_k$ if $1 \leqslant k \leqslant 8$, $\Lambda_{8+j} = t\partial_j + x_j\partial_t$ if $1 \leqslant j \leqslant 3$, $\Lambda^a = \Lambda_1^{a_1}\ldots\Lambda_{11}^{a_{11}}$ if $a = (a_1,\ldots,a_{11}) \in \mathbb{N}^{11}$. The following result is an immediate consequence of Proposition 3.1.

Proposition 3.2. *One can find $\varepsilon_0, C_a, C_{k\delta} > 0$ such that the following holds: if $\varepsilon \leqslant \varepsilon_0$, then*

(1) $|\Lambda^a\theta_1| + |\Lambda^a w_1| \leqslant C_a\varepsilon\sigma_-^{-2}\sigma_+^{-1}$,
(2) $(|\partial\Lambda^a\theta_1| + |\partial\Lambda^a w_1|)(x,t) \leqslant C_a\varepsilon\langle t\rangle^{-4}$ *if* $|x| \leqslant \frac{t}{2} + M + 1$,
(3) $|X^k w_1(x,t)| \leqslant C_{k\delta}\varepsilon\langle t\rangle^{-4+\delta}$ *if* $\delta < 1$ *and* $|x| \leqslant M\langle t\rangle^\delta$.

Now set $z_1(x,t) = \varepsilon z_0(x,\varepsilon)$, $\theta_2 = \theta - \theta_1$, $w_2 = w - w_1$, $z_2 = z - z_1$. It follows from (3.6)–(3.11) that

$$\partial_t\theta_2 + \nabla\cdot w_2 = -(w\cdot\nabla\theta - w_1\cdot\nabla\theta_1) + (\theta\nabla\cdot w - \theta_1\nabla\cdot w_1), \tag{3.12}$$

$$\partial_t w_2 + \nabla\theta_2 = -(w\cdot\nabla w - w_1\cdot\nabla w_1) + (\theta\nabla\theta - \theta_1\nabla\theta_1) - (1-\theta)z\nabla\theta, \tag{3.13}$$

$$\partial_t z_2 = -(w_1+w_2)\nabla\cdot(z_1+z_2), \tag{3.14}$$

$$\theta_2(x,0) = 0, \tag{3.15}$$

$$w_2(x,0) = 0, \tag{3.16}$$

$$z_2(x,0) = 0. \tag{3.17}$$

Notice that (3.12)–(3.14) is just the system (19)–(21) of [5], in which $C_1 = -1$ now.

Set $(\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4,\Gamma_5) = (\partial_t,\partial_1,\partial_2,\partial_3,X)$ and define $\Gamma^a = \Gamma_1^{a_1}\ldots\Gamma_5^{a_5}$ if $a = (a_1,\ldots,a_5) \in \mathbb{N}^5$. When $n \in \mathbb{N}$, set $E_n(t) = \sum_{|a|\leqslant n}(\|\Gamma^a\theta_2(t)\|^2 + \|\Gamma^a w_2(t)\|^2 + \|\Gamma^a z_2(t)\|^2)$. Fix a function $\psi : (0,1) \to \mathbb{R}^+$ with $\varepsilon\psi(\varepsilon)$ bounded and $\psi(\varepsilon) \to +\infty$ as $\varepsilon \xrightarrow{>} 0$, such that ψ has a strictly positive lower bound. Theorem 2.1 is of course an immediate consequence of the following result:

Theorem 3.1. *Fix $\mu \in \mathbb{N}$. If $\varepsilon_0 > 0$ is small and $\varepsilon \leqslant \varepsilon_0$,* (3.12)–(3.17) *has a unique solution $(\theta_2, w_2, z_2) \in C^\infty(\mathbb{R}^3 \times [0,+\infty))$, with $E_\mu^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$.*

If $\xi \in \mathbb{R}$, write $[\xi] = \sup\{\xi' \in \mathbb{Z},\ \xi' \leqslant \xi\}$. Theorem 3.1 will be a consequence of the following two results:

Theorem 3.2. *Fix $\mu, m \in \mathbb{N}$ with $[(m+5)/2] \leqslant \mu \leqslant m$. One can find C, $\varepsilon_0 > 0$ such that the following holds: if $\varepsilon \leqslant \varepsilon_0$, $T > 0$, and (θ_2, w_2, z_2) is a C^∞ solution of* (3.12)–(3.17) *when $x \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, with $E_\mu^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$, then $E_m^{1/2}(t) \leqslant C\varepsilon^2\langle t\rangle^{C\varepsilon}$ if $0 \leqslant t \leqslant T$.*

Theorem 3.2 will be used for proving the second result.

Theorem 3.3. *Fix $\mu \in \mathbb{N}$ with $\mu \geqslant 6$. One can find $\tilde{\varepsilon}_0 > 0$ such that the following holds: if $\varepsilon \leqslant \tilde{\varepsilon}_0$, $T > 0$, and (θ_2, w_2, z_2) is a C^∞ solution of* (3.12)–(3.17) *when $x \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, with $E_\mu^{1/2}(t) \leqslant \varepsilon^2 \psi(\varepsilon)$, then $E_\mu^{1/2}(t) \leqslant \frac{1}{2}\varepsilon^2\psi(\varepsilon)$ if $0 \leqslant t \leqslant T$.*

Replacing θ, w, z by $\theta_1 + \theta_2, w_1 + w_2, z_1 + z_2$ in (3.12)–(3.14), we remark that (3.12)–(3.14) is a symmetrizable hyperbolic (with respect to t) system, with unknowns (θ_2, w_2, z_2), if $|z| \leqslant \frac{1}{2}$ (see also (4.18) below). See e.g. [11] for properties of symmetrizable hyperbolic systems. Hence a standard continuation argument using Theorem 3.3 shows that Theorem 3.1 holds.

The proof of Theorem 3.2 resembles very much that of Theorem 4 of [5] with some refinements; in particular we shall make use of better decay estimates (contained in Proposition 4.2 below). For Theorem 3.3, these better decay estimates will also be needed; besides, in the isentropic region, improved decay estimates will be obtained by introducing a potential and exploiting null conditions; here estimates from [17] will be very important. In fact, the strategy of proof of Theorem 3.1, based on controlling a "high" norm (in Theorem 3.2) and a "low" norm (in Theorem 3.3) is similar to that used in [17] (see also [7]).

In subsequent estimates we shall denote by C various strictly positive constants which are independent of ε, T, and of the point where the estimates are being considered. When convenient, we shall write C_ν to stress the dependence of C on some parameter ν. In this paper there are a number of results requiring ε to be small; when using them, we shall tacitly assume that ε is as small as needed for applying them.

4. Preliminary estimates

Write $\sigma(x) = \langle |x| \rangle$. If $n \in \mathbb{N} \setminus \{0\}$, set $Q_n(t) = \sum_{|a| \leqslant n-1} (\|\sigma_-(t)\nabla\Gamma^a\theta_2(t)\| + \|\sigma_-(t)\partial_t\Gamma^a w_2(t)\| + \|\sigma_-(t)\nabla \cdot \Gamma^a w_2(t)\|)$. Define $\widetilde{Q}_n(t) = Q_n(t) + E_{n-1}^{1/2}(t)$ if $n \in \mathbb{N}$ and $n \geqslant 1$, $\widetilde{\widetilde{Q}}_n(t) = Q_n(t) + E_{n-2}^{1/2}(t)$ if $n \in \mathbb{N}$ and $n \geqslant 2$. The following estimates will play a crucial role in the sequel.

Lemma 4.1.

(1) $\|\sigma_-(t)\nabla v\| \leqslant C(\|\sigma_-(t)\nabla \cdot v\| + \|v\|)$ *if* $v \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ *and* $\nabla \times v = 0$,
(2) $|\sigma\Gamma^b w_2(t)| + |\sigma\Gamma^b\theta_2(t)| + |\sigma\Gamma^b z_2(t)| \leqslant C_b E_{|b|+2}^{1/2}(t)$,
(3) $|\sigma\sigma_-(t)\nabla\Gamma^b\theta_2(t)| \leqslant C_b Q_{|b|+3}(t)$,
(4) $|\sigma\sigma_-(t)\nabla\Gamma^b w_2(t)| \leqslant C_b \widetilde{Q}_{|b|+3}(t)$,
(5) $|\sigma\sigma_-^{1/2}(t)\Gamma^b\theta_2(t)| \leqslant C_b \widetilde{\widetilde{Q}}_{|b|+2}(t)$,
(6) $|\sigma\sigma_-^{1/2}(t)\Gamma^b w_2(t)| \leqslant C_b \widetilde{Q}_{|b|+2}(t)$.

Proof. (1) is the 3D analogue of formula (6.7) of [15] and is proved in exactly the same way. (2)–(4) are just Lemma 2 of [5]; they follow immediately from Lemma 3.3 of [16], its proof, and the proof of Proposition 3.3 of [16]. Finally, it also follows from Lemma 3.3 of [16], its proof, and the proof of Proposition 3.3 of [16] that

$$\sigma(x)\sigma_-^{1/2}(x,t)|v(x)| \leqslant C\bigg(\sum_{|a| \leqslant 1} \|\sigma_-(t)\nabla Y^a v\| + \sum_{|d| \leqslant 2} \|\Omega^d v\|\bigg)$$

if $v \in C_0^\infty(\mathbb{R}^3)$, where $Y^a = Y_1^{a_1} \dots Y_6^{a_6}$ if $a = (a_1, \dots, a_6) \in \mathbb{N}^6$, with $(Y_1, \dots, Y_6) = (\partial_1, \partial_2, \partial_3, \Omega_1, \Omega_2, \Omega_3)$, and $\Omega^d = \Omega_1^{d_1}\Omega_2^{d_2}\Omega_3^{d_3}$ if $d = (d_1, d_2, d_3) \in \mathbb{N}^3$. (5) and (6) follow easily. □

In the sequel, we shall make repeated use of the system obtained when applying $\partial^\alpha(X+1)^k$ to (3.12)–(3.13), that is, if we put $\Gamma^a = \partial^\alpha X^k$,

$$\partial_t\Gamma^a\theta_2 + \nabla \cdot \Gamma^a w_2 = h_0^a, \tag{4.1}$$

$$\partial_t\Gamma^a w_2 + \nabla\Gamma^a\theta_2 = h^a, \tag{4.2}$$

where

$$h_0^a = \sum_{1\leqslant j\leqslant 6} \tau_j^a, \quad h^a = \sum_{7\leqslant j\leqslant 13} \tau_j^a, \quad \text{with } \tau_1^a = -\sum_{b\leqslant a} \binom{a}{b} \Gamma^b w_1 \cdot \nabla \Gamma^{a-b}\theta_2,$$

$$\tau_2^a = -\sum_{b\leqslant a} \binom{a}{b} \Gamma^b w_2 \cdot \nabla \Gamma^{a-b}\theta_1, \quad \tau_3^a = -\sum_{b\leqslant a} \binom{a}{b} \Gamma^b w_2 \cdot \nabla \Gamma^{a-b}\theta_2,$$

$$\tau_4^a = \sum_{b\leqslant a} \binom{a}{b} \Gamma^b \theta_1 \nabla \cdot \Gamma^{a-b} w_2, \quad \tau_5^a = \sum_{b\leqslant a} \binom{a}{b} \Gamma^b \theta_2 \nabla \cdot \Gamma^{a-b} w_1,$$

$$\tau_6^a = \sum_{b\leqslant a} \binom{a}{b} \Gamma^b \theta_2 \nabla \cdot \Gamma^{a-b} w_2, \quad \tau_7^a = -\sum_{b\leqslant a} \binom{a}{b} \Gamma^b w_1 \cdot \nabla \Gamma^{a-b} w_2, \quad \tau_8^a = -\sum_{b\leqslant a} \binom{a}{b} \Gamma^b w_2 \cdot \nabla \Gamma^{a-b} w_1,$$

$$\tau_9^a = -\sum_{b\leqslant a} \binom{a}{b} \Gamma^b w_2 \cdot \nabla \Gamma^{a-b} w_2, \quad \tau_{10}^a = \sum_{b\leqslant a} \binom{a}{b} \Gamma^b \theta_1 \nabla \Gamma^{a-b}\theta_2, \quad \tau_{11}^a = \sum_{b\leqslant a} \binom{a}{b} \Gamma^b \theta_2 \nabla \Gamma^{a-b}\theta_1,$$

$$\tau_{12}^a = \sum_{b\leqslant a} \binom{a}{b} \Gamma^b \theta_2 \nabla \Gamma^{a-b}\theta_2, \quad \tau_{13}^a = -\partial^\alpha (X+1)^k \big((1-\theta) z \nabla\theta\big).$$

The term τ_{13}^a is created by the non-constant entropy. When $r \neq t$, it follows from (31) of [5] and the computations leading to (34) of [5] that (4.1), (4.2) imply that

$$\begin{aligned}
\nabla \Gamma^a \theta_2 = {} & -\frac{t}{t^2-r^2} X \Gamma^a w_2 + \frac{t}{t^2-r^2} (\Omega \times \Gamma^a w_2) - X \Gamma^a \theta_2 \frac{x}{t^2-r^2} \\
& + \frac{1}{t^2-r^2} (x \times \Omega \Gamma^a \theta_2) + \frac{t}{t^2-r^2} h_0^a x + \frac{t^2}{t^2-r^2} h^a, \qquad (4.3)\\
\nabla \cdot \Gamma^a w_2 = {} & -\frac{t}{t^2-r^2} X \Gamma^a \theta_2 - \frac{1}{t^2-r^2} X \Gamma^a w_2 \cdot x \\
& + \frac{1}{t^2-r^2} (\Omega \times \Gamma^a w_2) \cdot x + \frac{t^2}{t^2-r^2} h_0^a + \frac{t}{t^2-r^2} h^a \cdot x. \qquad (4.4)
\end{aligned}$$

In particular, (4.3) and (4.4) are useful for proving the following proposition:

Proposition 4.1. *Fix $n \in \mathbb{N}$ with $n \geqslant 4$. One can find $\eta, C, \varepsilon_0 > 0$ such that the following holds. If $\varepsilon \leqslant \varepsilon_0$, $T > 0$, and (θ_2, w_2, z_2) is a C^∞ solution of (3.12)–(3.17) when $x \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, with $E_{[n/2]+2}^{1/2}(t) \leqslant \eta$, then $Q_n(t) \leqslant C(E_n^{1/2}(t) + \frac{\varepsilon^2}{\langle t\rangle^3})$ if $0 \leqslant t \leqslant T$.*

Actually, Proposition 4.1 can be obtained by adapting the proof of Proposition 3 of [5]. A proof is given in Appendix B at the end of the present paper.

If $f : [0, +\infty) \to \mathbb{R}$, we put $I(f)(r) = \frac{1}{r^2}\int_0^r s^2 f(s)\,\mathrm{d}s$ when this makes sense. We have the following lemma (cf. Lemma 2.1(c) of [1] in the 2D case).

Lemma 4.2. *If $\alpha \in \mathbb{N}^3 \setminus \{0\}$, one can find $C_\alpha > 0$ such that the following holds. For all C^∞ functions $f : [0, +\infty) \to \mathbb{R}$ with $f^{(l)}(0) = 0$ when l is odd and $l \leqslant |\alpha| - 2$, and for all $R > 0$, one has*:

$$\sup_{0<|x|\leqslant R} \left| \partial_x^\alpha \left(I(f)(r) \frac{x}{r} \right) \right| \leqslant C_\alpha \sum_{j\leqslant |\alpha|-1} \sup_{0\leqslant s\leqslant R} \big| f^{(j)}(s) \big|.$$

Proof. Put $F(s) = \frac{I(f)(s)}{s}$ if $s > 0$. It is easily checked by induction that for some functions $h_{\alpha,n}(x)$, C^∞ for $x \neq 0$ and positively homogeneous of degree n,

$$\partial_x^\alpha \big(F(r)\big) = \sum_{1\leqslant k\leqslant |\alpha|} F^{(k)}(r) h_{\alpha,k-|\alpha|}(x) \quad \text{if } \alpha \neq 0. \qquad (4.5)$$

On the other hand, another induction using integration by parts shows that

$$F^{(k)}(r) = \frac{I(f_k)(r)}{r^{k+1}} \tag{4.6}$$

if $f_k(s) = s^k f^{(k)}(s)$ (cf. Lemma 2.1(b) of [1] in the 2D case). Now, using (4.6) and integrating by parts, we find that $\frac{F^{(k)}(r)}{r^{|\alpha|-k}} = \frac{f^{(k-1)}(r)}{r^{|\alpha|-(k-1)}} - \frac{k+2}{r^{|\alpha|+3}} \int_0^r s^{k+1} f^{(k-1)}(s)\,\mathrm{d}s$. Combining this with (4.5), we obtain that

$$\left|\partial_x^\alpha\big(F(r)\big)\right| \leqslant C_\alpha \sum_{k \leqslant |\alpha|-1} \Big(\sup_{0 \leqslant s \leqslant r} \left|f^{(k)}(s)\right|\Big) r^{k-|\alpha|} \quad \text{if } \alpha \neq 0,$$

whence

$$\left|\partial_x^\alpha\big(F(r)x\big)\right| \leqslant C_\alpha \sum_{k \leqslant |\alpha|-1} \Big(\sup_{0 \leqslant s \leqslant r} \left|f^{(k)}(s)\right|\Big) r^{k+1-|\alpha|} \quad \text{if } \alpha \neq 0. \tag{4.7}$$

As a special case, we obtain from (4.7), using the Taylor formula if $|\alpha| \geqslant 2$, that with C_α independent of $R > 0$,

$$\sup_{0<|x| \leqslant R} \left|\partial_x^\alpha\big(F(r)x\big)\right| \leqslant C_\alpha \sup_{0 \leqslant s \leqslant R} \left|f^{(|\alpha|-1)}(s)\right| \quad \text{if } \alpha \neq 0,$$
$$\text{and if furthermore } f^{(j)}(0) = 0 \text{ for all } j \leqslant |\alpha| - 2 \text{ in case } |\alpha| \geqslant 2. \tag{4.8}$$

We can now complete the proof of Lemma 4.2. Lemma 4.2 is just (4.8) if $|\alpha| = 1$. When $|\alpha| \geqslant 2$, write $f(s) = \sum_{l \leqslant |\alpha|-2} \frac{f^{(l)}(0)}{l!} s^l + g(s)$. Applying (4.8) with f replaced by g, we find that with C_α independent of $\widetilde{R} > 0$:

$$\sum_{0<|x| \leqslant \widetilde{R}} \left|\partial_x^\alpha\left(I(g)(r)\frac{x}{r}\right)\right| \leqslant C_\alpha \sup_{0 \leqslant s \leqslant \widetilde{R}} \left|f^{(|\alpha|-1)}(s)\right|. \tag{4.9}$$

On the other hand, if we put $\tilde{f}_l(s) = \frac{f^{(l)}(0)}{l!} s^l$, we obtain:

$$\sup_{|x| \leqslant 1} \left|\partial_x^\alpha\left(I\big(\tilde{f}_l\big)(r)\frac{x}{r}\right)\right| \leqslant C_l \left|f^{(l)}(0)\right| \quad \text{if } l \text{ is even}. \tag{4.10}$$

Now $f^{(l)}(0) = 0$ if l is odd and $l \leqslant |\alpha| - 2$; so using (4.9) and (4.10), we obtain Lemma 4.2 if $R \leqslant 1$. If now $R > 1$, we just combine the case $R \leqslant 1$ of Lemma 4.2 with (4.7). The proof of Lemma 4.2 is complete. □

For a function $g(x,t)$, put $|g(t)|_- = \sup_{r \leqslant t/2+M+1} |g(x,t)|$. We shall use several times the following decay estimates.

Proposition 4.2. *Fix $\mu \in \mathbb{N}$ with $\mu \geqslant 4$. One can find η, ε_0, $C > 0$ such that the following holds. If $\varepsilon \leqslant \varepsilon_0$, $T > 0$, and (θ_2, w_2, z_2) is a C^∞ solution of* (3.12)–(3.17) *when $x \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, with $E_\mu^{1/2}(t) \leqslant \varepsilon^2 \psi(\varepsilon)$, and if $\widetilde{Q}_{|a|+2}(t) \leqslant \eta \langle t \rangle^{1/2}$ in case $|a| \geqslant \mu - 1$, then*

(1) $|\nabla \Gamma^a \theta_2(t)|_- + |\nabla \cdot \Gamma^a w_2(t)|_- \leqslant C\chi_{|a|}(t)$ *if $|a| \leqslant 2\mu - 3$, where $\chi_k(t) = \frac{Q_{k+3}(t)}{\langle t \rangle^{3/2}} + \frac{\varepsilon^2}{\langle t \rangle^4}$ (so $\chi_{|a|}(t) \leqslant C\frac{\varepsilon^2 \psi(\varepsilon)}{\langle t \rangle^{3/2}}$ if $|a| \leqslant \mu - 3$).*
(2) $|\partial_t \Gamma^a \theta_2(t)|_- + |\partial_t \Gamma^a w_2(t)|_- \leqslant C\chi_{|a|}(t)$ *if $|a| \leqslant 2\mu - 3$.*
(3) $|\Gamma^a w_2(t)|_- \leqslant C\chi_{|a|-1}(t)$ *if $|a| \leqslant 2\mu - 2$ and $a_2 + a_3 + a_4 \neq 0$.*
(4) $|\Gamma^a w_2(x,t)| \leqslant Cr\chi_{|a|}(t)$ *if $|a| \leqslant 2\mu - 3$, $a_2 + a_3 + a_4 = 0$, and $r \leqslant \frac{t}{2} + M + 1$.*

Proof. (1) If t has a fixed upper bound, Proposition 4.2(1), (2) follows at once from the Sobolev injection theorem, so when proving Proposition 4.2(1), (2), we may and shall assume that t is so large that $(t^2 - r^2)\langle t \rangle^{-2}$ has a strictly positive lower bound if $r \leqslant \frac{t}{2} + M + 1$. Set $g_a(t) = |\nabla \Gamma^a \theta_2(t)|_- + |\nabla \cdot \Gamma^a w_2(t)|_-$, $G_a(t) = \sum_{b \leqslant a} g_b(t)$, $G_a^\star(t) = \sum_{|b| \leqslant |a|} g_b(t)$. With the help of Lemma 4.1(5), (6), it readily follows from (4.3), (4.4) that

$$g_a(t) \leqslant C\left(\frac{\widetilde{Q}_{|a|+3}(t)}{\langle t \rangle^{3/2}} + \left|h_0^a(t)\right|_- + \left|h^a(t)\right|_-\right). \tag{4.11}$$

Using Proposition 3.2(1), (2), Lemmas 4.1(5), (6) and 4.2, we easily check that

$$\sum_{1\leqslant j\leqslant 6}\left|\tau_j^a(t)\right|_- \leqslant C\left(\frac{\varepsilon}{\langle t\rangle^{9/2}}\widetilde{Q}_{|a|+2}(t)+\left(\frac{\varepsilon}{\langle t\rangle^3}+\frac{\widetilde{Q}_{|a|+2}(t)}{\langle t\rangle^{1/2}}\right)G_a(t)\right), \tag{4.12}$$

$$\sum_{7\leqslant j\leqslant 12}\left|\tau_j^a(t)\right|_- \leqslant C\left(\frac{\varepsilon}{\langle t\rangle^{9/2}}\widetilde{Q}_{|a|+2}(t)+\left(\frac{\varepsilon}{\langle t\rangle^3}+\frac{\widetilde{Q}_{|a|+2}(t)}{\langle t\rangle^{1/2}}\right)G^\star_{|a|}(t)\right). \tag{4.13}$$

If $\Gamma^a=\partial^\alpha X^k$, we have $|\partial^\alpha(X+1)^k(z\nabla\theta)(t)|_-\leqslant C\sum_{b\leqslant a;\, i,j=1,2}|(\Gamma^b z_i\nabla\Gamma^{a-b}\theta_j)(t)|_-$, which can be bounded above by $C((\varepsilon+E^{1/2}_{|a|+2}(t))\frac{\varepsilon}{\langle t\rangle^4}+\varepsilon G_a(t)+\sum_{b\leqslant a}E^{1/2}_{|b|+2}(t)g_{a-b}(t))$ thanks to Proposition 3.2(2) and the Sobolev injection theorem. Now $\langle t\rangle^{-1/2}\widetilde{Q}_{|a|+2}(t)$ is bounded (if $|a|\leqslant\mu-2$, this is implied by the assumption $E_\mu^{1/2}(t)\leqslant\varepsilon^2\psi(\varepsilon)$ and by Proposition 4.1). Hence, writing $\theta=\theta_1+\theta_2$, and using Proposition 3.2(1) and Lemma 4.1(5), we obtain that $|\partial^{\alpha'}X^{k'}\theta(t)|_-\leqslant C$ if $\alpha'\leqslant\alpha$, $k'\leqslant k$. Summing up, we obtain that

$$\left|\tau_{13}^a(t)\right|_-\leqslant C\left(\left(\varepsilon+E^{1/2}_{|a|+2}(t)\right)\frac{\varepsilon}{\langle t\rangle^4}+\varepsilon G_a(t)+\sum_{b\leqslant a}E^{1/2}_{|b|+2}(t)g_{a-b}(t)\right). \tag{4.14}$$

Assume first that $|a|\leqslant\mu-2$. Then $\widetilde{Q}_{|a|+2}(t)\leqslant C\varepsilon^2\psi(\varepsilon)$ thanks to Proposition 4.1, and using (4.12)–(4.14), we readily obtain that

$$\left|h_0^a(t)\right|_-+\left|h^a(t)\right|_-\leqslant C\left(\frac{\varepsilon^2}{\langle t\rangle^4}+\varepsilon G^\star_{|a|}(t)\right).$$

(1) follows easily if we use (4.11). If now $\mu-1\leqslant|a|\leqslant 2\mu-3$, we have $E^{1/2}_{|b|+2}g_{a-b}\leqslant E^{1/2}_\mu G_a$ if $b\leqslant a$ and $|b|\leqslant\mu-2$; and if $b\leqslant a$ and $|b|\geqslant\mu-1$, then $|a-b|\leqslant\mu-2$, but in this case we already know that $g_{a-b}\leqslant C\chi_{|a-b|}$, so $E^{1/2}_{|b|+2}g_{a-b}\leqslant CE^{1/2}_{|b|+2}\chi_{|a-b|}$. From (4.12)–(4.14) we obtain that

$$\left|h_0^a(t)\right|_-+\left|h^a(t)\right|_-\leqslant C\left((\varepsilon+\eta)G^\star_{|a|}(t)+\frac{\varepsilon^2}{\langle t\rangle^4}+\frac{\varepsilon}{\langle t\rangle^{3/2}}\left(\widetilde{Q}_{|a|+2}(t)+E^{1/2}_{|a|+2}(t)\right)\right).$$

(1) follows easily if we use (4.11).
(2) has already been proved for t small. For t large, it follows at once from (4.1), (4.2), (1), and the estimates of $|h_0^a(t)|_-+|h^a(t)|_-$ obtained in the proof of (1).
(3) Writing $\Gamma^a=\partial_x^\alpha\partial_t^n X^k$, applying Lemma 4.2 with $f(r)=(\nabla\cdot\partial_t^n X^k w_2)(x,t)$, and using (1), we obtain at once (3).
(4) If $a_2+a_3+a_4=0$, set $f(r,t)=\nabla\cdot\Gamma^a w_2(x,t)$; then $\Gamma^a w_2(x,t)=(\frac{1}{r^2}\int_0^r s^2 f(s,t)\,\mathrm{d}s)\frac{x}{r}$, so (4) follows at once from (1). The proof of Proposition 4.2 is complete. □

As a first use of Proposition 4.2 we are going to show that, when the assumptions of Theorem 3.2 (or 3.3) are fulfilled and ε is small, there is a fixed ball of $\mathbb{R}^3$ outside which the flow is isentropic whatever t. More precisely:

Proposition 4.3. *Assume that $\mu\geqslant 4$. One can find $\varepsilon_0>0$ such that the following holds: if $\varepsilon\leqslant\varepsilon_0$, if $T>0$ and if (θ_2,w_2,z_2) is a C^∞ solution to* (3.12)–(3.17) *when $x\in\mathbb{R}^3$, $0\leqslant t\leqslant T$, such that $E_\mu^{1/2}(t)\leqslant\varepsilon^2\psi(\varepsilon)$, then $z_2(x,t)=0$ if $|x|\geqslant M+1$.*

Proof. Define $W_j(r,t)$ by the relation $w_j(x,t)=W_j(r,t)\frac{x}{r}$ and put $W=W_1+W_2$. Let $r_1(t)$ be defined by $r_1'(t)=W(r_1(t),t)$, $r_1(0)=M$. Of course it follows from (3.8) that $z(x,t)=0$ if $|x|\geqslant r_1(t)$. By Proposition 3.2(1), Lemma 4.1(4) and Proposition 4.1, we have $|\partial_r W(r,t)|\leqslant\frac{C\varepsilon}{\langle t\rangle}$; so $|W(r,t)|\leqslant\frac{C\varepsilon}{\langle t\rangle}r$. But then a standard comparison argument shows that $r_1(t)\leqslant M\langle t\rangle^{C\varepsilon}$. Therefore if we fix $\delta\in(0,1)$ and take ε small enough, we have $r_1(t)\leqslant M\langle t\rangle^\delta$. But then $|W_1(r_1(t),t)|\leqslant\frac{C\varepsilon}{\langle t\rangle^{4-\delta}}$ by Proposition 3.2(3), and $|W_2(r_1(t),t)|\leqslant C\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$ with the help of Proposition 4.2(4). Hence $|W(r_1(t),t)|\leqslant\frac{C\varepsilon}{\langle t\rangle^{3/2-\delta}}$. Choosing $\delta<1/2$, we obtain that $|r_1(t)-M|\leqslant C\varepsilon$, which proves Proposition 4.3. □

In the proof of Theorems 3.2 and 3.3, we shall use the classical energy method. It follows from (4.1), (4.2), (3.14) that

$$(\partial_t + w\cdot\nabla)\Gamma^a\theta_2 + (1-\theta)\nabla\cdot\Gamma^a w_2 = \hat{h}_0^a, \tag{4.15}$$

$$\left(\frac{1}{1+z}\partial_t + \frac{w}{1+z}\cdot\nabla\right)\Gamma^a w_2 + (1-\theta)\nabla\Gamma^a\theta_2 = \frac{\hat{h}^a}{1+z}, \tag{4.16}$$

$$(\partial_t + w\cdot\nabla)\Gamma^a z_2 = \hat{g}^a, \tag{4.17}$$

where we assume, as we may, that $|z| \leqslant \frac{1}{2}$ (this is allowed since $E_2^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$). The functions $\hat{h}_0^a$, $\hat{h}^a$, $\hat{g}^a$ are defined as follows: $\hat{h}_0^a = \sum_{j\in\{2,5\}}\tau_j^a + \sum_{j\in\{1,3,4,6\}}\hat{\tau}_j^a$, $\hat{h}^a = \sum_{j\in\{8,11\}}\tau_j^a + \sum_{j\in\{7,9,10,12,13\}}\hat{\tau}_j^a$, where τ_j^a are the same as in h_0^a, h^a (from (4.1), (4.2)); if $a \neq 0$, $\hat{\tau}_j^a$ ($j \neq 13$) are defined as τ_j^a but with the supplementary condition that $b \neq 0$ in the sum; $\hat{\tau}_j^0 = 0$ if $j \neq 13$; finally $\hat{\tau}_{13}^a = \tau_{13}^a + (1-\theta)z\nabla\Gamma^a\theta_2$. Also $\hat{g}^a = \sum_{i,j\in\{1,2\}}\hat{g}_{ij}^a$, with $\hat{g}_{i1}^a = -\sum_{b\leqslant a}\binom{a}{b}\Gamma^b w_i\cdot\nabla\Gamma^{a-b}z_1$; when $a \neq 0$, $\hat{g}_{i2}^a = -\sum_{0\neq b\leqslant a}\binom{a}{b}\Gamma^b w_i\cdot\nabla\Gamma^{a-b}z_2$, and $\hat{g}_{i2}^0 = 0$.

If $\xi \in \mathbb{R}^3$, let $(\xi^{(l)})_{1\leqslant l\leqslant 3}$ be its components in the canonical basis of $\mathbb{R}^3$. Define the (1×5) matrices $\zeta^a = \mathrm{tr}(\Gamma^a\theta_2\ (\Gamma^a w_2)^{(l)}_{1\leqslant l\leqslant 3}\ \Gamma^a z_2)$, $F^a = \mathrm{tr}\ (\hat{h}_0^a\ (\frac{\hat{h}^a}{1+z})^{(l)}_{1\leqslant l\leqslant 3}\ \hat{g}^a)$, where tr means transpose. Define also the (5×5) matrices A_j, $0 \leqslant j \leqslant 3$, with elements $A_j^{k,l}$, in the following way: $A_0^{1,1} = A_0^{5,5} = 1$, $A_0^{k,k} = \frac{1}{1+z}$ if $2 \leqslant k \leqslant 4$, $A_0^{k,l} = 0$ otherwise; if $1 \leqslant j \leqslant 3$, $A_j^{1,1} = A_j^{5,5} = w^{(j)}$, $A_j^{k,k} = \frac{w^{(j)}}{1+z}$ if $2 \leqslant k \leqslant 4$, $A_j^{1,j+1} = A_j^{j+1,1} = 1-\theta$, $A_j^{k,l} = 0$ otherwise. Then (4.15)–(4.17) can be written

$$A_0\partial_t\zeta^a + \sum_{1\leqslant j\leqslant 3} A_j\partial_j\zeta^a = F^a, \tag{4.18}$$

which is a symmetric hyperbolic system. Taking the (pointwise) Euclidean scalar product (in $\mathbb{R}^5$) with ζ^a, integrating over $\mathbb{R}_x^3$ and using the symmetry of A_j, $0 \leqslant j \leqslant 3$, we obtain (writing ∂_0 for ∂_t when convenient):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle A_0\zeta^a, \zeta^a\rangle(t) = \langle F^a, \zeta^a\rangle(t) + \frac{1}{2}\sum_{0\leqslant j\leqslant 3}\langle(\partial_j A_j)\zeta^a, \zeta^a\rangle(t); \tag{4.19}$$

here and in the sequel, for two functions $f(x,t)$, $\tilde{f}(x,t)$ valued in $\mathbb{R}^l$, we write: $\langle f, \tilde{f}\rangle(t) = \int_{\mathbb{R}^3} f(x,t)\cdot\tilde{f}(x,t)\,\mathrm{d}x$, where now $\cdot$ is the usual scalar product in $\mathbb{R}^l$ ($l = 5$ in (4.19)). Now it follows from (3.12)–(3.17) that, for some $C_m^\star > 0$ independent of ε,

$$E_m^{1/2}(0) \leqslant C_m^\star\varepsilon^2. \tag{4.20}$$

Set $P_j^a = \langle\tau_j^a, \Gamma^a\theta_2\rangle$ if $j = 2,5$, $P_j^a = \langle\hat{\tau}_j^a, \Gamma^a\theta_2\rangle$ if $j = 1,3,4,6$, $P_j^a = \langle\frac{\tau_j^a}{1+z}, \Gamma^a w_2\rangle$ if $j = 8,11$, $P_j^a = \langle\frac{\hat{\tau}_j^a}{1+z}, \Gamma^a w_2\rangle$ if $j = 7,9,10,12,13$, $\widehat{P}_{ij}^a = \langle\hat{g}_{ij}^a, \Gamma^a z_2\rangle$ if $i,j = 1,2$. Then

$$\langle F^a, \zeta^a\rangle = \sum_{1\leqslant j\leqslant 13} P_j^a + \sum_{1\leqslant i,j\leqslant 2}\widehat{P}_{ij}^a. \tag{4.21}$$

Also observe that

$$\left|\sum_{0\leqslant j\leqslant 3}\partial_j A_j\right| \leqslant C\big(|\nabla\theta| + |\nabla\cdot w|\big). \tag{4.22}$$

5. Proof of Theorem 3.2

Throughout this section, we shall suppose that the assumptions of Theorem 3.2 hold, in particular that $m \geqslant \mu \geqslant 4$, and that ε is small, so that in particular $|z| \leqslant \frac{1}{2}$. Theorem 3.2 will be obtained by refining slightly the estimates in the proof of Theorem 4 of [5]. First we have:

$$\left|P_j^a(t)\right| \leqslant C\varepsilon\frac{E_m(t)}{\langle t\rangle} \quad \text{if } |a| \leqslant m \text{ and } j \in \{1,2,4,5,7,8,10,11\}. \tag{5.1}$$

Let us bound $|P_j^a(t)|$ if $j \in \{3,6,9,12\}$. Let us consider $|P_9^a(t)|$ since the other ones can be treated in a similar (and even somewhat simpler) way. Assume that $|a| \leqslant m$ and $|a-b| \leqslant m-1$. If $|b| \leqslant |a-b|$, we have $\|\sigma_+(t)(\Gamma^b w_2 \cdot \nabla \Gamma^{a-b} w_2)(t)\| \leqslant C|\sigma \Gamma^b w_2(t)| \|\sigma_-(t)\nabla \Gamma^{a-b} w_2(t)\| \leqslant C E^{1/2}_{[m/2]+2}(t) \widetilde{Q}_m(t)$ by Lemma 4.1(1), (2). If $|b| > |a-b|$, we have $\|\sigma_+(t)(\Gamma^b w_2 \cdot \nabla \Gamma^{a-b} w_2)(t)\| \leqslant \|\Gamma^b w_2(t)\| |\sigma_+(t)\nabla \Gamma^{a-b} w_2(t)| \leqslant C E_m^{1/2}(t) \widetilde{Q}_{[(m+5)/2]}(t)$ by Lemma 4.1(4). Using Proposition 4.1, we find that $\|\sigma_+(t)(\Gamma^b w_2 \cdot \nabla \Gamma^{a-b} w_2)(t)\| \leqslant C(E_\mu^{1/2}(t) E_m^{1/2}(t) + \frac{\varepsilon^2}{\langle t\rangle^3} E_m^{1/2}(t))$. Arguing similarly for $j = 3, 6, 12$, we finally obtain that

$$\left|P_j^a(t)\right| \leqslant C\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle} E_m(t) \quad \text{if } |a| \leqslant m \text{ and } j \in \{3,6,9,12\}. \tag{5.2}$$

If $\Gamma^a = \partial^\alpha X^k$ with $|a| \leqslant m$, we can write $\hat{\tau}_{13}^a = F_1 + F_2 + F_3$, where $F_1 = -\partial^\alpha (X+1)^k (z\nabla\theta) + z\nabla\Gamma^a\theta$, $F_2 = \partial^\alpha (X+1)^k(\theta z \nabla\theta) - \theta z \nabla \Gamma^a \theta$, $F_3 = -(1-\theta) z \nabla \Gamma^a \theta_1$. If $i,j = 1,2$ and $0 \neq b \leqslant a$, set $\mathcal{L}_{ijb}(t) = \|\sigma_+(t)(\Gamma^b z_i \nabla \Gamma^{a-b}\theta_j)(t)\|$. We have $\mathcal{L}_{11b}(t) \leqslant C\frac{\varepsilon^2}{\langle t\rangle^3}$ with the help of Proposition 3.2(2), $\mathcal{L}_{12b}(t) \leqslant C\varepsilon Q_m(t)$. Using Propositions 4.3 and 3.2(2), we obtain that $\mathcal{L}_{21b}(t) \leqslant C\frac{\varepsilon}{\langle t\rangle^3} E_m^{1/2}(t)$. If $|b| \leqslant |a-b|$,

$$\mathcal{L}_{22b}(t) \leqslant C\left|\sigma \Gamma^b z_2(t)\right| \left\|\sigma_-(t)\nabla\Gamma^{a-b}\theta_2(t)\right\| \leqslant C E^{1/2}_{[m/2]+2}(t) Q_m(t)$$

with the help of Lemma 4.1(2). If $|b| > |a-b|$,

$$\mathcal{L}_{22b}(t) \leqslant \left\|\Gamma^b z_2(t)\right\| \left|\sigma_+(t)\nabla\Gamma^{a-b}\theta_2(t)\right| \leqslant C E_m^{1/2}(t) Q_{[(m+5)/2]}(t)$$

with the help of Lemma 4.1(3) (one could also use Proposition 4.2(1), but this would not improve (5.3) below). Making use of Proposition 4.1, we find that

$$\left\|\sigma_+(t) F_1(t)\right\| \leqslant C\varepsilon\left(E_m^{1/2}(t) + \frac{\varepsilon}{\langle t\rangle^3}\right). \tag{5.3}$$

Now $F_2 = F_4 - \theta F_1$, where $F_4 = \partial^\alpha (X+1)^k(\theta z\nabla\theta) - \theta\partial^\alpha(X+1)^k(z\nabla\theta)$. Set $F_{4j\alpha'k'} = \partial^{\alpha'} X^{k'} \theta_j \partial^{\alpha''}(X+1)^{k''}(z\nabla\theta)$ if $\alpha' + \alpha'' = \alpha$, $k' + k'' = k$, $|\alpha'| + k' > 0$. Using Propositions 4.3 and 3.2(1), and the arguments of the proof of (5.3), we obtain that

$$\left\|\sigma_+(t) F_{41\alpha'k'}(t)\right\| \leqslant C\frac{\varepsilon^2}{\langle t\rangle^3}\left(E_m^{1/2}(t) + \frac{\varepsilon}{\langle t\rangle^3}\right).$$

If $|\alpha'| + k' \leqslant |\alpha''| + k''$, we have $|\partial^{\alpha'} X^{k'} \theta_2(x,t)| \leqslant C\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle^{1/2}}$ by Lemma 4.1(5). Therefore, using the arguments of the proof of (5.3), we obtain that

$$\left\|\sigma_+(t) F_{42\alpha'k'}(t)\right\| \leqslant C\frac{\varepsilon^3\psi(\varepsilon)}{\langle t\rangle^{1/2}}\left(E_m^{1/2}(t) + \frac{\varepsilon}{\langle t\rangle^3}\right) \quad \text{if } |\alpha'| + k' \leqslant |\alpha''| + k''.$$

Put $\tilde{\sigma}_+(x,t) = (1 + |x|^2 + t^2)^{1/2}$. If $|\alpha'| + k' > |\alpha''| + k''$, we have:

$$\left\|\sigma_+(t) F_{42\alpha'k'}(t)\right\| \leqslant C\left\|\partial^{\alpha'} X^{k'}\theta_2(t)\right\| \left|\tilde{\sigma}_+(t)\partial^{\alpha''}(X+1)^{k''}(z\nabla\theta)(t)\right|.$$

Estimating the second factor in the right-hand side of the last inequality by means of the Sobolev injection theorem and using the arguments of the proof of (5.3), we find that

$$\left\|\sigma_+(t) F_{42\alpha'k'}(t)\right\| \leqslant C\varepsilon E_m^{1/2}(t)\left(E_\mu^{1/2}(t) + \frac{\varepsilon}{\langle t\rangle^3}\right) \quad \text{if } |\alpha'| + k' > |\alpha''| + k''.$$

Collecting estimates, we now conclude that

$$\left\|\sigma_+(t) F_4(t)\right\| \leqslant C\varepsilon^2\left(E_m^{1/2}(t) + \frac{\varepsilon}{\langle t\rangle^{7/2}}\right).$$

Finally, using Propositions 4.3 and 3.2(2), we obtain that

$$\left\|\sigma_+(t) F_3(t)\right\| \leqslant C\frac{\varepsilon^2}{\langle t\rangle^3}.$$

Summing up, we have proved that

$$\big\|\sigma_+(t)\hat{\tau}^a_{13}(t)\big\| \leqslant C\varepsilon\bigg(E_m^{1/2}(t)+\frac{\varepsilon}{\langle t\rangle^3}\bigg), \tag{5.4}$$

from which it follows that

$$\big|P^a_{13}(t)\big| \leqslant \frac{C\varepsilon}{\langle t\rangle}E_m^{1/2}(t)\bigg(E_m^{1/2}(t)+\frac{\varepsilon}{\langle t\rangle^3}\bigg). \tag{5.5}$$

Using Propositions 4.3 and 3.2(1), we easily find that

$$\big|\widehat{P}^a_{1j}(t)\big| \leqslant C\frac{\varepsilon}{\langle t\rangle^3}E_m^{1/2}(t)\big(E_m^{1/2}(t)+\varepsilon\big), \quad j=1,2. \tag{5.6}$$

To study $\widehat{P}^a_{2j}(t)$, we just keep refining the corresponding estimates of [5]. Put $\nu_j^{ab}=\Gamma^b w_2\cdot\nabla\Gamma^{a-b}z_j$. Assume first that $b_l\neq 0$ for some $l\leqslant 4$. Then $\Gamma^b=\partial_{l-1}\Gamma^d$ with $|d|=|b|-1$. We then have with the help of Proposition 4.3:

$$\big|\nu_j^{ab}(t)\big| \leqslant \frac{C}{\langle t\rangle}\sigma_-(x,t)\big|\partial_{l-1}\Gamma^d w_2(x,t)\cdot\nabla\Gamma^{a-b}z_j(x,t)\big|. \tag{5.7}$$

Using (5.7) and, if $l\neq 1$, Lemma 4.1(5), we obtain after an application of Proposition 4.1:

$$\big\|\nu_1^{ab}(t)\big\| \leqslant C\frac{\varepsilon}{\langle t\rangle}\bigg(E_m^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\bigg) \quad \text{if } b_l\neq 0 \text{ for some } l\leqslant 4,$$

$$\big\|\nu_2^{ab}(t)\big\| \leqslant C\frac{E_\mu^{1/2}(t)}{\langle t\rangle}\bigg(E_m^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\bigg) \quad \text{if } b_l\neq 0 \text{ for some } l\leqslant 4 \text{ and } |a-b|<|b|.$$

If $b_l\neq 0$ for some $l\leqslant 4$ and $|b|\leqslant|a-b|$, we write:

$$\big\|\nu_2^{ab}(t)\big\| \leqslant \Big(\sup_{|x|\leqslant M+1}\big|\partial_{l-1}\Gamma^d w_2(x,t)\big|\Big)\big\|\nabla\Gamma^{a-b}z_2(t)\big\| \leqslant \frac{C}{\langle t\rangle}\bigg(E_\mu^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\bigg)E_m^{1/2}(t),$$

if we make use of Lemma 4.1(4) if $l\geqslant 2$, of the Sobolev injection theorem if $l=1$, and of Proposition 4.1. So we obtain that

$$\big\|\nu_2^{ab}(t)\big\| \leqslant \frac{C}{\langle t\rangle}\bigg(E_\mu^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\bigg)E_m^{1/2}(t) \quad \text{if } b_l\neq 0 \text{ for some } l\leqslant 4.$$

To continue, we recall in the following lemma some useful estimates contained in Lemma 4 of [5] (part (2) of the following lemma is a trivial refinement of Lemma 4(ii) of [5] and is proved in the same way).

Lemma 5.1. *Fix $m\in\mathbb{N}$. One can find $C>0$ and, for all $R>0$, also $C_R>0$ such that the following holds: if $T>0$ and (θ_2,w_2,z_2) is a C^∞ solution of (3.12)–(3.17) if $x\in\mathbb{R}^3$, $0\leqslant t\leqslant T$, then*

(1) $|X^k w_2(x,t)|\leqslant C\frac{r}{\langle t\rangle}\widetilde{Q}_{k+3}(t)$ *if $k\in\mathbb{N}$ and $k\leqslant m-3$.*
(2) $\langle t\rangle^K\|\prod_{1\leqslant k\leqslant K}X^{m_k}W_2(t)\|_{L^2(\{x\in\mathbb{R}^3,|x|\leqslant R\})}\leqslant C_R\prod_{1\leqslant k\leqslant K}Q_{m_k+1}(t)$ *if $m_k\leqslant m-1$ for $1\leqslant k\leqslant K$, $K=1,2$, and $t\geqslant 0$.*

Let us go back to the estimates of ν_j^{ab}. If now $\Gamma^b=X^{|b|}$ with $|b|\leqslant m-1$, we use Proposition 4.3, Lemma 5.1(2), Proposition 4.1 and the Sobolev injection theorem to check that

$$\big\|\nu_1^{ab}(t)\big\| \leqslant C\frac{\varepsilon}{\langle t\rangle}\bigg(E_m^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\bigg);$$

$$\big\|\nu_2^{ab}(t)\big\| \leqslant \frac{C}{\langle t\rangle}E_\mu^{1/2}(t)\bigg(E_m^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\bigg) \quad \text{if } |a-b|<|b|.$$

If $0<|b|<|a-b|$, we use Proposition 4.3, Lemma 5.1(1) and Proposition 4.1 to obtain that

$$\big\|\nu_2^{ab}(t)\big\| \leqslant \frac{C}{\langle t\rangle}\bigg(E_\mu^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\bigg)E_m^{1/2}(t) \quad \text{if } 0<|b|<|a-b|.$$

One could repeat the same reasoning if $|b| = |a-b|$ provided that $\mu \geqslant [\frac{m}{2}] + 3$. However we only assume that $\mu \geqslant [\frac{m+5}{2}]$. Therefore we may argue as follows. Set $\tilde{\sigma}_-(x,t) = \lambda(t - \lambda(r))$, where $\lambda(s) = (1+s^2)^{1/2}$. We have $\langle t\rangle \|X^{|b|} w_2(t)\|_{L^6(\{x\in\mathbb{R}^3,\ |x|\leqslant M+1\})} \leqslant C\|\tilde{\sigma}_-(t) X^{|b|} w_2(t)\|_{L^6(\mathbb{R}^3)}$, which can be bounded by $C\|\nabla(\tilde{\sigma}_- X^{|b|} w_2)(t)\|$ by a standard Sobolev estimate. Hence

$$\big\|X^{|b|} w_2(t)\big\|_{L^6(\{x\in\mathbb{R}^3,\ |x|\leqslant M+1\})} \leqslant C\frac{\widetilde{Q}_{|b|+1}(t)}{\langle t\rangle}.$$

On the other hand, $\|\nabla \Gamma^{a-b} z_2(t)\|_{L^3(\mathbb{R}^3)} \leqslant C\sum_{1\leqslant j\leqslant 2} \|\nabla^j \Gamma^{a-b} z_2(t)\|$ by another standard Sobolev estimate. So if $\Gamma^b = X^{|b|}$, we conclude with the help of the Hölder inequality that

$$\big\|v_2^{ab}(t)\big\| \leqslant \frac{C}{\langle t\rangle}\bigg(E_\mu^{1/2}(t) + \frac{\varepsilon^2}{\langle t\rangle^3}\bigg)E_\mu^{1/2}(t) \quad \text{if } |b| = |a-b|.$$

Set $\Phi_j = X^m z_2 \nabla z_j$, $\varphi_j = \langle X^m w_2, \Phi_j\rangle$. Collecting the estimates obtained so far, we have proved the following:

$$\begin{aligned}
&\text{If } |a|\leqslant m \text{ and } \Gamma^a \neq X^m, \quad \text{then } \big|\widehat{P}_{21}^a(t)\big| \leqslant C\frac{\varepsilon}{\langle t\rangle} E_m^{1/2}(t)\bigg(E_m^{1/2}(t) + \frac{\varepsilon^2}{\langle t\rangle^3}\bigg),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \big|\widehat{P}_{22}^a(t)\big| \leqslant C\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle} E_m(t),\\
&\text{If } \Gamma^a = X^m, \quad \text{then } \big|\widehat{P}_{21}^a(t) + \varphi_1(t)\big| \leqslant C\frac{\varepsilon}{\langle t\rangle} E_m^{1/2}(t)\bigg(E_m^{1/2}(t) + \frac{\varepsilon^2}{\langle t\rangle^3}\bigg),\\
&\qquad\qquad\qquad\qquad \big|\widehat{P}_{22}^a(t) + \varphi_2(t)\big| \leqslant C\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle} E_m(t).
\end{aligned} \tag{5.8}$$

As in [5], we write $\varphi_j(t) = \frac{\mathrm{d}}{\mathrm{d}t}(tI_j(t)) - I_j(t) + J_j(t) + N_j(t)$, with

$$\begin{aligned}
I_j(t) &= \int \big(X^{m-1} w_2 \cdot \Phi_j\big)(x,t)\,\mathrm{d}x,\\
J_j(t) &= \int \big(r\partial_r X^{m-1} w_2 \cdot \Phi_j\big)(x,t)\,\mathrm{d}x,\\
N_j(t) &= -t\int \big(X^{m-1} w_2 \cdot \partial_t \Phi_j\big)(x,t)\,\mathrm{d}x.
\end{aligned}$$

Here and in the sequel, the integrals with respect to dx are taken over $\mathbb{R}^3$ if the domain is not mentioned. We have:

$$\big|I_j(t)\big| + \big|J_j(t)\big| \leqslant C\frac{\lambda_j(\varepsilon)}{\langle t\rangle} E_m^{1/2}(t)\bigg(E_m^{1/2}(t) + \frac{\varepsilon^2}{\langle t\rangle^3}\bigg),$$

where $\lambda_1(\varepsilon) = \varepsilon$, $\lambda_2(\varepsilon) = \varepsilon^2\psi(\varepsilon)$. This follows at once from (62) and (63) of [5] if we make use of Proposition 4.1 of the present paper. As in [5] we write $N_j = N_{j1} + N_{j2}$, where

$$\begin{aligned}
N_{j1}(t) &= -t\int \big(X^{m-1} w_2 \cdot \partial_t X^m z_2 \nabla z_j\big)(x,t)\,\mathrm{d}x,\\
N_{j2}(t) &= -t\int \big(X^{m-1} w_2 \cdot X^m z_2 \partial_t \nabla z_j\big)(x,t)\,\mathrm{d}x.
\end{aligned}$$

Since $\partial_t z_1 \equiv 0$, we have $N_{12} \equiv 0$. It follows at once from (65) of [5] and Proposition 4.1 of the present paper that

$$\big|N_{22}(t)\big| \leqslant C\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle} E_m^{1/2}(t)\bigg(E_m^{1/2}(t) + \frac{\varepsilon^2}{\langle t\rangle^3}\bigg).$$

As in [5], we write (if $\Gamma^a = X^m$) $\tilde{g}_1 = \hat{g}_{11}^a$, $\tilde{g}_2 = \hat{g}_{12}^a$, $\tilde{g}_3 = \hat{g}_{21}^a$, $\tilde{g}_4 = \hat{g}_{22}^a$, $N_{j1} = \sum_{1\leqslant k\leqslant 5} N_{j1k}$, with

$$\begin{aligned}
N_{j11}(t) &= t\int \big(X^{m-1} w_2 \cdot \nabla z_j\big)(x,t)\big(w\cdot\nabla X^m z_2\big)(x,t)\,\mathrm{d}x,\\
N_{j1k}(t) &= -t\int \big(X^{m-1} w_2 \cdot \nabla z_j\big)(x,t)\tilde{g}_{k-1}(x,t)\,\mathrm{d}x \quad \text{if } 2\leqslant k\leqslant 5.
\end{aligned}$$

By Propositions 3.2(3) and 4.3,

$$\left|X^k w_1(x,t)\right| \leqslant C_k \frac{\varepsilon}{\langle t\rangle^4} \quad \text{on } \operatorname{supp} \nabla z_j. \tag{5.9}$$

It follows easily with the help of (5.9), Propositions 4.2(4), (1) and 4.1, and Lemma 5.1 that

$$\left|N_{j11}(t)\right| \leqslant C \frac{\varepsilon\lambda_j(\varepsilon)}{\langle t\rangle^{3/2}} E_m^{1/2}(t)\left(E_m^{1/2}(t) + \frac{\varepsilon^2}{\langle t\rangle^3}\right).$$

This is just the estimate (67) of [5] with $p_3(t)$ replaced by $\frac{\varepsilon}{\langle t\rangle^{3/2}}$, and it can be proved in the same way with the help of Proposition 4.1 of the present paper. Also,

$$\left|N_{j1k}(t)\right| \leqslant C \frac{\varepsilon\lambda_j(\varepsilon)}{\langle t\rangle^5}\left(E_m(t) + \varepsilon E_m^{1/2}(t) + \frac{\varepsilon^3}{\langle t\rangle^3}\right) \quad \text{if } k=2,3,$$

which can be proved by repeating the proof of (68) of [5], using (5.9), Lemma 5.1(2), and Proposition 4.1 of the present paper. If $k=4,5$, we write $N_{j1k} = \sum_{k-4\leqslant l\leqslant m}\binom{m}{l} N_{j1kl}$, where

$$N_{j1kl}(t) = t\int \left(X^{m-1}w_2\cdot\nabla z_j\right)(x,t)\left(X^l w_2\cdot\nabla X^{m-l} z_{k-3}\right)(x,t)\,\mathrm{d}x.$$

Assume that $l\leqslant m-1$. If $m-l\leqslant l+3$ (so that $m-l+1\leqslant\mu$), we write:

$$\left|N_{j1kl}(t)\right| \leqslant t\left\|\left(X^{m-1}W_2X^lW_2\right)(t)\right\|_{L^2(\{x\in\mathbb{R}^3,\ |x|\leqslant M+1\})}\left|\nabla z_j(t)\right|\left\|\nabla X^{m-l}z_{k-3}(t)\right\|.$$

Hence, using Lemma 5.1(2), Proposition 4.1 and the Sobolev injection theorem, we obtain that $|N_{j1kl}| \leqslant C\frac{\varepsilon\lambda_j(\varepsilon)}{\langle t\rangle}(E_m^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3})^2$. If now $m-l>l+3$, we write:

$$\left|N_{j1kl}(t)\right| \leqslant t\left\|X^{m-1}w_2(t)\right\|_{L^2(\{x\in\mathbb{R}^3,\ |x|\leqslant M+1\})}\left(\sup_{|x|\leqslant M+1}\left|X^l w_2(x,t)\right|\right)\left|\nabla z_j(t)\right|\left\|\nabla X^{m-l}z_{k-3}(t)\right\|.$$

We can bound the right-hand side of this last inequality by using Lemma 5.1(2), Propositions 4.1, 4.2(4) and the Sobolev injection theorem, and obtain that

$$\left|N_{j1kl}(t)\right| \leqslant C\frac{\varepsilon^2\psi(\varepsilon)\lambda_j(\varepsilon)}{\langle t\rangle^{3/2}}\left(E_m^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\right)\left(E_m^{1/2}(t)+\varepsilon\right)$$

if $m-l>l+3$. Summing up, we conclude that

$$\left|N_{j1kl}(t)\right| \leqslant C\frac{\varepsilon\lambda_j(\varepsilon)}{\langle t\rangle}\left(E_m(t)+\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle^{1/2}}E_m^{1/2}(t)+\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^{7/2}}\right) \quad \text{if } k=4,5 \text{ and } l\leqslant m-1.$$

Arguing exactly as for (69) of [5], we write:

$$N_{j1km}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(tI_{jk}(t)\right)+P_{jk}(t),$$

where $I_{jk}(t) = t\int (X^{m-1}W_2)^2(r,t)(\partial_r z_j\partial_r z_{k-3})(x,t)\,\mathrm{d}x$; for I_{jk}, P_{jk} exactly the same estimates as in [5] can be used (with the help of Proposition 4.1 of the present paper and with $p_3(t)$ replaced by $\frac{\varepsilon}{\langle t\rangle^{3/2}}$), so that

$$\left|I_{jk}(t)\right| \leqslant C\frac{\varepsilon\lambda_j(\varepsilon)}{\langle t\rangle}\left(E_m^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\right)^2,$$

$$\left|P_{jk}(t)\right| \leqslant C\frac{\varepsilon\lambda_j(\varepsilon)}{\langle t\rangle}\left(E_m^{1/2}(t)+\frac{\varepsilon^2}{\langle t\rangle^3}\right)^2.$$

Collecting estimates, we obtain that

$$\left|N_{j1}(t)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t\sum_{k=4,5}I_{jk}(t)\right)\right| \leqslant C\frac{\varepsilon\lambda_j(\varepsilon)}{\langle t\rangle}\left(E_m(t)+\frac{\varepsilon}{\langle t\rangle^{1/2}}E_m^{1/2}(t)+\frac{\varepsilon^3}{\langle t\rangle^{7/2}}\right).$$

So finally,

$$\left|\varphi_j(t)-\frac{\mathrm{d}}{\mathrm{d}t}\left(t\left(I_j(t)+\frac{1}{2}\sum_{k=4,5}I_{jk}(t)\right)\right)\right|\leqslant C\left(\frac{\lambda_j(\varepsilon)}{\langle t\rangle}E_m(t)+\frac{\varepsilon^2\lambda_j(\varepsilon)}{\langle t\rangle^{3/2}}E_m^{1/2}(t)+\frac{\varepsilon^4\lambda_j(\varepsilon)}{\langle t\rangle^{9/2}}\right).$$

We can therefore complete (5.8) with the following conclusion:

$$\text{If } \Gamma^a=X^m,\quad \text{then } \left|\widehat{P}^a_{2j}(t)-\frac{\mathrm{d}}{\mathrm{d}t}H_{1j}(t)\right|\leqslant C\left(\frac{\lambda_j(\varepsilon)}{\langle t\rangle}E_m(t)+\frac{\varepsilon^2\lambda_j(\varepsilon)}{\langle t\rangle^{3/2}}E_m^{1/2}(t)+\frac{\varepsilon^4\lambda_j(\varepsilon)}{\langle t\rangle^{9/2}}\right),$$
$$\text{where } \left|H_{1j}(t)\right|\leqslant C\left(\lambda_j(\varepsilon)E_m(t)+\frac{\varepsilon^2\lambda_j(\varepsilon)}{\langle t\rangle^3}E_m^{1/2}(t)+\frac{\varepsilon^5\lambda_j(\varepsilon)}{\langle t\rangle^6}\right)\text{ and } H_{1j}(0)=0. \tag{5.10}$$

Using (4.21), (5.1), (5.2), (5.5), (5.6), (5.8), (5.10), we conclude that

$$\sum_{|a|\leqslant m}\left\langle F^a,\zeta^a\right\rangle(t)=\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+H_2(t),\quad \text{where}$$
$$\left|H_1(t)\right|\leqslant C\left(\varepsilon E_m(t)+\frac{\varepsilon^3}{\langle t\rangle^3}E_m^{1/2}(t)+\frac{\varepsilon^6}{\langle t\rangle^6}\right),\ H_1(0)=0,\ \text{and}$$
$$\left|H_2(t)\right|\leqslant C\left(\frac{\varepsilon}{\langle t\rangle}E_m(t)+\frac{\varepsilon^2}{\langle t\rangle^{3/2}}E_m^{1/2}(t)+\frac{\varepsilon^5}{\langle t\rangle^{9/2}}\right). \tag{5.11}$$

On the other hand, using (4.22), Proposition 3.2(1) and Lemma 4.1(3), (4), we obtain that

$$\sum_{|a|\leqslant m}\left|\left\langle\sum_{0\leqslant j\leqslant 3}(\partial_jA_j)\zeta^a,\zeta^a\right\rangle\right|\leqslant C\frac{\varepsilon}{\langle t\rangle}E_m(t). \tag{5.12}$$

Put $N=\sum_{|a|\leqslant m}\langle A_0\zeta^a,\zeta^a\rangle-2H_1$. From (4.19) and (5.11), it follows that

$$\left|N'(t)\right|\leqslant 2\left|H_2(t)\right|+\sum_{|a|\leqslant m}\left|\left\langle\sum_{0\leqslant j\leqslant 3}(\partial_jA_j)\zeta^a,\zeta^a\right\rangle(t)\right|,$$

so by the estimate of $|H_2|$ in (5.11) and by (5.12), we obtain that

$$\left|N'(t)\right|\leqslant C\frac{\varepsilon}{\langle t\rangle}\left(E_m(t)+\frac{\varepsilon^4}{\langle t\rangle^{7/2}}\right)+C\frac{\varepsilon^2}{\langle t\rangle^{3/2}}E_m^{1/2}(t). \tag{5.13}$$

Now put $\widetilde{N}(t)=N(t)+\frac{\varepsilon^4}{\langle t\rangle^{7/2}}$. Using the estimate of H_1 from (5.11), we can easily check that, if ε is small,

$$\widetilde{C}_1\leqslant\frac{\widetilde{N}(t)}{E_m(t)+\varepsilon^4\langle t\rangle^{-7/2}}\leqslant\widetilde{C}_2, \tag{5.14}$$

for some fixed constants $\widetilde{C}_1,\widetilde{C}_2>0$. From (5.13) and (5.14) it follows that

$$\widetilde{N}'(t)\leqslant C_1\frac{\varepsilon}{\langle t\rangle}\widetilde{N}(t)+C_2\frac{\varepsilon^2}{\langle t\rangle^{3/2}}\widetilde{N}^{1/2}(t). \tag{5.15}$$

Put $\varphi(t)=\langle t\rangle^{-C_1\varepsilon}\widetilde{N}(t)$. Then (5.15) gives that $\varphi'(t)\leqslant C_2\varepsilon^2\langle t\rangle^{-(3+C_1\varepsilon)/2}\varphi^{1/2}(t)$. Hence

$$\left(\varphi^{1/2}(t)\right)'\leqslant\frac{1}{2}C_2\varepsilon^2\langle t\rangle^{-(3+C_1\varepsilon)/2},$$

so using (4.20), we obtain that $\varphi^{1/2}(t)\leqslant C\varepsilon^2$. This completes the proof of Theorem 3.2 thanks to (5.14). □

6. Proof of Theorem 3.3

For proving Theorem 3.3, we shall need better decay estimates than for Theorem 3.2. If $r \leqslant \frac{t}{2} + M + 1$, we shall use Proposition 4.2; if $r \geqslant \frac{t}{2} + M + 1$, the "null condition" $\gamma = -1$ will play a crucial role. Put $D_-(t) = \{x \in \mathbb{R}^3, |x| \leqslant \frac{t}{2} + M + 1\}$, $D_+(t) = \{x \in \mathbb{R}^3, |x| \geqslant \frac{t}{2} + M + 1\}$. If $f(x,t)$, $\tilde{f}(x,t)$ are $\mathbb{R}^l$-valued, write $\langle f, \tilde{f}\rangle_\pm(t) = \int_{D_\pm(t)} f(x,t) \cdot \tilde{f}(x,t)\,\mathrm{d}x$, $\|f(t)\|_\pm = \langle f, f\rangle_\pm^{1/2}(t)$, $|f(t)|_\pm = \sup_{x \in D_\pm(t)} |f(x,t)|$ (the notation $|\cdot(t)|_-$ has already been used in Section 4).

The next proposition gives estimates in $D_-(t)$.

Proposition 6.1. *Fix $\mu \in \mathbb{N}$ with $\mu \geqslant 6$. One can find $C > 0$, and, for each $\delta > 0$, also $\varepsilon_0 > 0$, such that the following holds: if $\varepsilon \leqslant \varepsilon_0$, $T > 0$, and (θ_2, w_2, z_2) is a C^∞ solution of* (3.12)–(3.17) *when $x \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, with $E_\mu^{1/2}(t) \leqslant \varepsilon^2 \psi(\varepsilon)$, then*

(1) $\sum_{|a| \leqslant \mu} |\langle F^a, \zeta^a\rangle_-(t) - \frac{\mathrm{d}}{\mathrm{d}t} D^a(t)| \leqslant C \frac{\varepsilon^4 \psi(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$, *where $D^a \in C^\infty[0,T]$, $D^a \equiv 0$ if $\mu \geqslant 7$ or if $\mu = 6$ and $\Gamma^a \neq X^\mu$, $|D^a(t)| \leqslant C \frac{\varepsilon^5 \psi(\varepsilon)}{\langle t\rangle^{1/2-\delta}}$ and $D^a(0) = 0$ if $\mu = 6$ and $\Gamma^a = X^\mu$;*

(2) $\sum_{|a| \leqslant \mu} |\langle \sum_{0 \leqslant j \leqslant 3} (\partial_j A_j)\zeta^a, \zeta^a\rangle_-(t)| \leqslant C \frac{\varepsilon^5 \psi^2(\varepsilon)}{\langle t\rangle^{3/2}}$.

Proof. Taking ε small, we shall assume, as we may, that $|z| \leqslant \frac{1}{2}$. The constants C will be independent of δ and we shall suppose that $\varepsilon \leqslant \varepsilon_0$ for some small ε_0 which may depend on δ.

So assume that $|a| \leqslant \mu$. Let τ_j^a, $\hat{\tau}_j^a$, $\hat{g}_{ij}^a$, P_j^a, $\widehat{P}_{ij}^a$ be as in Sections 4 and 5. Define P_{j-}^a, $j \leqslant 12$, as P_j^a, but with $\langle,\rangle$ replaced everywhere by $\langle,\rangle_-$. Of course, thanks to Proposition 4.3,

$$\langle F^a, \zeta^a\rangle_- = \sum_{1 \leqslant j \leqslant 12} P_{j-}^a + P_{13}^a + \sum_{1 \leqslant i,j \leqslant 2} \widehat{P}_{ij}^a. \tag{6.1}$$

Now $\|\hat{\tau}_1^a(t)\|_- \leqslant C \sum_{0 \neq b \leqslant a} |\Gamma^b w_1(t)|_- \|\nabla \Gamma^{a-b}\theta_2(t)\|_-$, so making use of Propositions 3.2(1) and 4.1, we obtain that

$$|P_{1-}^a(t)| \leqslant C \frac{\varepsilon}{\langle t\rangle^3} \frac{Q_\mu(t)}{\langle t\rangle} E_\mu^{1/2}(t) \leqslant C \frac{\varepsilon^5 \psi^2(\varepsilon)}{\langle t\rangle^4}.$$

$\|\tau_2^a(t)\|_- \leqslant C \sum_{b \leqslant a} \|\Gamma^b w_2(t)\|_- |\nabla \Gamma^{a-b}\theta_1(t)|_-$, so making use of Proposition 3.2(2), we obtain that

$$|P_{2-}^a(t)| \leqslant C E_\mu^{1/2}(t) \frac{\varepsilon}{\langle t\rangle^4} E_\mu^{1/2}(t) \leqslant C \frac{\varepsilon^5 \psi^2(\varepsilon)}{\langle t\rangle^4}.$$

$\|\hat{\tau}_3^a(t)\|_- \leqslant C \sum_{0 \neq b \leqslant a} |\Gamma^b w_2(t)|_- \|\nabla \Gamma^{a-b}\theta_2(t)\|_-$, so making use of Lemma 4.1(6), Proposition 4.1, and Theorem 3.2 with $m = \mu + 2$, we obtain that

$$|P_{3-}^a(t)| \leqslant C \frac{\widetilde{Q}_{\mu+2}(t)}{\langle t\rangle^{1/2}} \frac{Q_\mu(t)}{\langle t\rangle} E_\mu^{1/2}(t) \leqslant C \frac{\varepsilon^6 \psi^2(\varepsilon)}{\langle t\rangle^{3/2-\delta}}.$$

$\|\hat{\tau}_4^a(t)\|_- \leqslant C \sum_{0 \neq b \leqslant a} |\Gamma^b \theta_1(t)|_- \|\nabla \cdot \Gamma^{a-b} w_2(t)\|_-$, so making use of Propositions 3.2(1) and 4.1, we obtain that

$$|P_{4-}^a(t)| \leqslant C \frac{\varepsilon}{\langle t\rangle^3} \frac{Q_\mu(t)}{\langle t\rangle} E_\mu^{1/2}(t) \leqslant C \frac{\varepsilon^5 \psi^2(\varepsilon)}{\langle t\rangle^4}.$$

$\|\tau_5^a(t)\|_- \leqslant C \sum_{b \leqslant a} \|\Gamma^b \theta_2(t)\|_- |\nabla \cdot \Gamma^{a-b} w_1(t)|_-$, so making use of Proposition 3.2(2), we obtain that

$$|P_{5-}^a(t)| \leqslant C E_\mu^{1/2}(t) \frac{\varepsilon}{\langle t\rangle^4} E_\mu^{1/2}(t) \leqslant C \frac{\varepsilon^5 \psi^2(\varepsilon)}{\langle t\rangle^4}.$$

$\|\hat{\tau}_6^a(t)\|_- \leqslant C \sum_{0 \neq b \leqslant a} |\Gamma^b \theta_2(t)|_- \|\nabla \cdot \Gamma^{a-b} w_2(t)\|_-$, so making use of Lemma 4.1(5), Proposition 4.1, and Theorem 3.2 with $m = \mu + 2$, we obtain that

$$|P_{6-}^a(t)| \leqslant C \frac{\widetilde{Q}_{\mu+2}(t)}{\langle t\rangle^{1/2}} \frac{Q_\mu(t)}{\langle t\rangle} E_\mu^{1/2}(t) \leqslant C \frac{\varepsilon^6 \psi^2(\varepsilon)}{\langle t\rangle^{3/2-\delta}}.$$

$\|\hat{\tau}_7^a(t)\|_- \leqslant C\sum_{0\neq b\leqslant a}|\Gamma^b w_1(t)|_-\|\nabla\Gamma^{a-b}w_2(t)\|_-$, so making use of Proposition 3.2(1), Lemma 4.1(1), and Proposition 4.1, we obtain that

$$\left|P_{7-}^a(t)\right| \leqslant C\frac{\varepsilon}{\langle t\rangle^3}\frac{\widetilde{Q}_\mu(t)}{\langle t\rangle}E_\mu^{1/2}(t) \leqslant C\frac{\varepsilon^5\psi^2(\varepsilon)}{\langle t\rangle^4}.$$

$\|\tau_8^a(t)\|_- \leqslant C\sum_{b\leqslant a}\|\Gamma^b w_2(t)\|_-|\nabla\Gamma^{a-b}w_1(t)|_-$, so making use of Proposition 3.2(2), we obtain that

$$\left|P_{8-}^a(t)\right| \leqslant CE_\mu^{1/2}(t)\frac{\varepsilon}{\langle t\rangle^4}E_\mu^{1/2}(t) \leqslant C\frac{\varepsilon^5\psi^2(\varepsilon)}{\langle t\rangle^4}.$$

$\|\hat{\tau}_9^a(t)\|_- \leqslant C\sum_{0\neq b\leqslant a}|\Gamma^b w_2(t)|_-\|\nabla\Gamma^{a-b}w_2(t)\|_-$, so making use of Lemma 4.1(6), (1), Proposition 4.1, and Theorem 3.2 with $m=\mu+2$, we obtain that

$$\left|P_{9-}^a(t)\right| \leqslant C\frac{\widetilde{Q}_{\mu+2}(t)}{\langle t\rangle^{1/2}}\frac{\widetilde{Q}_\mu(t)}{\langle t\rangle}E_\mu^{1/2}(t) \leqslant C\frac{\varepsilon^6\psi^2(\varepsilon)}{\langle t\rangle^{3/2-\delta}}.$$

$\|\hat{\tau}_{10}^a(t)\|_- \leqslant C\sum_{0\neq b\leqslant a}|\Gamma^b\theta_1(t)|_-\|\nabla\Gamma^{a-b}\theta_2(t)\|_-$, so making use of Propositions 3.2(1) and 4.1, we obtain that

$$\left|P_{10-}^a(t)\right| \leqslant C\frac{\varepsilon}{\langle t\rangle^3}\frac{Q_\mu(t)}{\langle t\rangle}E_\mu^{1/2}(t) \leqslant C\frac{\varepsilon^5\psi^2(\varepsilon)}{\langle t\rangle^4}.$$

$\|\tau_{11}^a(t)\|_- \leqslant C\sum_{b\leqslant a}\|\Gamma^b\theta_2(t)\|_-|\nabla\Gamma^{a-b}\theta_1(t)|_-$, so making use of Proposition 3.2(2), we obtain that

$$\left|P_{11-}^a(t)\right| \leqslant CE_\mu^{1/2}(t)\frac{\varepsilon}{\langle t\rangle^4}E_\mu^{1/2}(t) \leqslant C\frac{\varepsilon^5\psi^2(\varepsilon)}{\langle t\rangle^4}.$$

$\|\hat{\tau}_{12}^a(t)\|_- \leqslant C\sum_{0\neq b\leqslant a}|\Gamma^b\theta_2(t)|_-\|\nabla\Gamma^{a-b}\theta_2(t)\|_-$, so making use of Lemma 4.1(5), Proposition 4.1, and Theorem 3.2 with $m=\mu+2$,we obtain that

$$\left|P_{12-}^a(t)\right| \leqslant C\frac{\widetilde{\widetilde{Q}}_{\mu+2}(t)}{\langle t\rangle^{1/2}}\frac{Q_\mu(t)}{\langle t\rangle}E_\mu^{1/2}(t) \leqslant C\frac{\varepsilon^6\psi^2(\varepsilon)}{\langle t\rangle^{3/2-\delta}}.$$

(5.4) shows that

$$\left\|\sigma_+(t)\hat{\tau}_{13}^a(t)\right\| \leqslant C\varepsilon\left(E_\mu^{1/2}(t)+\frac{\varepsilon}{\langle t\rangle^3}\right) \quad \text{if } |a|\leqslant\mu.$$

Now $|P_{13}^a(t)| \leqslant \frac{C}{\langle t\rangle}\|\sigma_+(t)\hat{\tau}_{13}^a(t)\|\|\Gamma^a w_2(t)\|_{L^2(\{x\in\mathbb{R}^3,\ |x|\leqslant M+1\})}$. Hence, thanks to Lemma 4.1(6), Proposition 4.1, and Theorem 3.2 with $m=\mu+2$, it follows that

$$\left|P_{13}^a(t)\right| \leqslant \frac{C}{\langle t\rangle}\varepsilon^2\frac{\varepsilon^2}{\langle t\rangle^{1/2-\delta}} = C\frac{\varepsilon^4}{\langle t\rangle^{3/2-\delta}}.$$

Finally,

$$\left|\langle\hat{g}_{ij}^a,\Gamma^a z_2\rangle(t)\right| \leqslant C\sum_{\substack{b\leqslant a\\ b\neq 0 \text{ if } j=2}}\Psi_{ij}^{ab}(t), \quad \text{where } \Psi_{ij}^{ab}(t) = \Big(\sup_{|x|\leqslant M+1}\left|\Gamma^b w_i(x,t)\right|\Big)\left\|\nabla\Gamma^{a-b}z_j(t)\right\|\left\|\Gamma^a z_2(t)\right\|.$$

Let $\lambda_j(\varepsilon)$ be as in Section 5. Using Proposition 3.2(1), we conclude that

$$\left|\widehat{P}_{1j}^a(t)\right| \leqslant C\frac{\varepsilon^3\psi(\varepsilon)\lambda_j(\varepsilon)}{\langle t\rangle^3}.$$

Using Proposition 4.2, and Theorem 3.2 with $m=\mu+2$, we obtain that

$$\Psi_{2j}^{ab}(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)\lambda_j(\varepsilon)}{\langle t\rangle^{3/2-\delta}} \tag{6.2}$$

except when $b=a$ and $\Gamma^a=X^\mu$. To handle this last case if $\mu\geqslant 7$, we apply Proposition 4.2(4), and Theorem 3.2 with $m=\mu+3$, and find that (6.2) still holds. Summing up, we have in particular that

$$\left|\widehat{P}_{2j}^a(t)\right| \leqslant C\frac{\varepsilon^4\psi(\varepsilon)\lambda_j(\varepsilon)}{\langle t\rangle^{3/2-\delta}} \quad \text{if } \mu\geqslant 7, \text{ or if } \mu=6 \text{ and } \Gamma^a\neq X^6.$$

If now $\mu = 6$ and $\Gamma^a = X^6$, let φ_j be as in (5.8), but with m replaced by 6. By (6.2),

$$\left|\widehat{P}_{2j}^a(t) + \varphi_j(t)\right| \leqslant C\frac{\varepsilon^4\psi(\varepsilon)\lambda_j(\varepsilon)}{\langle t\rangle^{3/2-\delta}} \quad \text{if } \mu = 6 \text{ and } \Gamma^a = X^6. \tag{6.3}$$

We have $\varphi_j(t) = \frac{\mathrm{d}}{\mathrm{d}t}(tI_j(t)) - I_j(t) + J_j(t) + N_j(t)$, with I_j, J_j, N_j as in Section 5, but with m now replaced by 6. Using Propositions 4.2(4), (3) and 4.1, and Theorem 3.2 with $m = 8$, $\mu = 6$, we readily find that

$$\left|I_j(t)\right| + \left|J_j(t)\right| \leqslant C\frac{\varepsilon^4\psi(\varepsilon)\lambda_j(\varepsilon)}{\langle t\rangle^{3/2-\delta}}.$$

Now $\|(X - r\partial_r)X^6 z_2(t)\| \leqslant CE_7^{1/2}(t)$ and $|(X - r\partial_r)\nabla z_2(t)| \leqslant CE_4^{1/2}(t)$. Hence with the help of Theorem 3.2 with $m = 7$, $\mu = 6$, and of the Sobolev injection theorem, we find that for some $C > 0$ and any $\tilde{\delta} > 0$, $\|t\partial_t(X^6 z_2 \nabla z_j)(t)\| \leqslant C\varepsilon^2\psi(\varepsilon)\lambda_j(\varepsilon)\langle t\rangle^{\tilde{\delta}}$ if $\varepsilon \leqslant \varepsilon_0(\tilde{\delta})$. Hence

$$\left|N_j(t)\right| \leqslant C\varepsilon^2\psi(\varepsilon)\lambda_j(\varepsilon)\langle t\rangle^{\tilde{\delta}}\chi_5(t),$$

with χ_5 as in Proposition 4.2. Combining all this with (6.3), we find that

$$\left|\widehat{P}_{2j}^a(t) + \frac{\mathrm{d}}{\mathrm{d}t}\big(tI_j(t)\big)\right| \leqslant C\frac{\varepsilon^4\psi(\varepsilon)\lambda_j(\varepsilon)}{\langle t\rangle^{3/2-\delta}} \quad \text{if } \mu = 6 \text{ and } \Gamma^a = X^6.$$

If we collect the estimates we have obtained, Proposition 6.1(1) follows at once from (6.1).

(2) Using Propositions 3.2(2) and 4.2, we obtain that

$$\left|\nabla\theta(t)\right|_- + \left|\nabla\cdot w(t)\right|_- \leqslant C\frac{\varepsilon}{\langle t\rangle^{3/2}}.$$

Hence from (4.22) it follows that

$$\left|\left\langle \sum_{0\leqslant j\leqslant 3}(\partial_j A_j)\zeta^a, \zeta^a\right\rangle_-\right| \leqslant C\frac{\varepsilon}{\langle t\rangle^{3/2}}\big(\varepsilon^2\psi(\varepsilon)\big)^2 \quad \text{if } |a| \leqslant \mu.$$

This proves Proposition 6.1(2). □

Now we shall obtain estimates in $D_+(t)$.

Proposition 6.2. *Fix $\mu \in \mathbb{N}$ with $\mu \geqslant 6$. One can find $C > 0$, and, for each $\delta > 0$, also $\varepsilon_0 > 0$, such that the following holds: if $\varepsilon \leqslant \varepsilon_0$, $T > 0$, and (θ_2, w_2, z_2) is a C^∞ solution of* (3.12)–(3.17) *when $x \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, with $E_\mu^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$, then*

(1) $\sum_{|a|\leqslant\mu} |\langle F^a, \zeta^a\rangle_+(t)| \leqslant C\frac{\varepsilon^5\psi^2(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$,
(2) $\sum_{|a|\leqslant\mu} |\langle \sum_{0\leqslant j\leqslant 3}(\partial_j A_j)\zeta^a, \zeta^a\rangle_+(t)| \leqslant C\frac{\varepsilon^5\psi(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$.

Almost all the rest of this section is devoted to the proof of Proposition 6.2. At the very end of this section, we shall see that Theorem 3.3 follows from Propositions 6.1 and 6.2. From now on we shall fix $\mu \in \mathbb{N}$ with $\mu \geqslant 6$; the constants C will be independent of $\delta > 0$ and we shall assume that $\varepsilon \leqslant \varepsilon_0$ for some small ε_0 which may depend on δ. Proposition 6.2 will be proved with the help of potentials. Let (θ_2, w_2, z_2) be as in the assumptions of Proposition 6.2 and let (ρ, u, S) be the corresponding solution of (2.1)–(2.6). Let $r_1(t)$ be as in the proof of Proposition 4.3; $S = \bar{S}$ if $|x| \geqslant r_1(t)$. Let $V \in C^\infty(\{(x,t) \in \mathbb{R}^3 \times [0,T], |x| \geqslant r_1(t)\})$ be the potential vanishing for $|x| \geqslant \bar{c}t + M$ and satisfying the relations $\nabla V = u$, $\partial_t V = -\frac{1}{2}|u|^2 - h(\rho)$ if $|x| \geqslant r_1(t)$ and $0 \leqslant t \leqslant T$. Let V_1 be as in Section 3. When $|x| \geqslant r_1(t)$ and $0 \leqslant t \leqslant T$, put $V_2 = V - V_1$. If $(\lambda, y), (\tilde{\lambda}, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^3$, put $F_1(\lambda, y) = \frac{1}{2}(\lambda^2 - |y|^2)$, $F_2((\lambda, y), (\tilde{\lambda}, \tilde{y})) = \frac{1}{2}(\lambda\tilde{\lambda} - y \cdot \tilde{y})$. Also write $\xi = (\theta, w)$, $\xi_j = (\theta_j, w_j)$. It is easily checked that

$$\theta_1(x,t) = -\frac{1}{\bar{c}^2}\partial_t V_1\left(x, \frac{t}{\bar{c}}\right) + F_1(\xi_1)(x,t), \tag{6.4}$$

and that, when $|x| \geqslant r_1(t)$,

$$\theta(x,t) = -\frac{1}{\bar{c}^2}\partial_t V\left(x, \frac{t}{\bar{c}}\right) + F_1(\xi)(x,t), \tag{6.5}$$

$$\theta_2(x,t) = -\frac{1}{\bar{c}^2}\partial_t V_2\left(x, \frac{t}{\bar{c}}\right) + F_2(\xi_2, 2\xi_1+\xi_2)(x,t). \tag{6.6}$$

Also,

$$w_1(x,t) = \frac{1}{\bar{c}}\nabla V_1\left(x, \frac{t}{\bar{c}}\right), \tag{6.7}$$

and, when $|x| \geqslant r_1(t)$,

$$w(x,t) = \frac{1}{\bar{c}}\nabla V\left(x, \frac{t}{\bar{c}}\right), \tag{6.8}$$

$$w_2(x,t) = \frac{1}{\bar{c}}\nabla V_2\left(x, \frac{t}{\bar{c}}\right). \tag{6.9}$$

Note the following elementary identities which will be used many times in the sequel.

$$\begin{aligned}&\text{For all } a=(a_1,\dots,a_5)\in\mathbb{N}^5,\ \text{one can find } c_{ab}\in\mathbb{R}\ (\text{where } b=(b_1,\dots,b_5)\in\mathbb{N}^5),\\ &\quad\text{with } c_{ab}=0 \text{ if } b_j\neq a_j \text{ for some } j\leqslant 4,\ \text{such that,}\\ &\quad\text{for all } i\in\{0,1,2,3\},\quad \Gamma^a\partial_i = \sum_{b\leqslant a} c_{ab}\partial_i\Gamma^b.\end{aligned} \tag{6.10}$$

$$\begin{aligned}&\text{For all } a=(a_1,\dots,a_5)\in\mathbb{N}^5,\ \text{one can find } \hat{c}_{ab}\in\mathbb{R}\ (\text{where } b=(b_1,\dots,b_5)\in\mathbb{N}^5),\\ &\quad\text{with } \hat{c}_{ab}=0 \text{ if } b_j\neq a_j \text{ for some } j\leqslant 4,\ \text{such that,}\\ &\quad\text{for all } i\in\{0,1,2,3\},\qquad \partial_i\Gamma^a = \sum_{b\leqslant a} \hat{c}_{ab}\Gamma^b\partial_i.\end{aligned} \tag{6.11}$$

Notice also that, at each point,

$$\left|\Gamma^a\big(F_2(\xi_2, 2\xi_1+\xi_2)\big)\right| \leqslant C\Bigg(\sum_{\substack{|b|\leqslant[|a|/2]\\ b\leqslant a}}\left|\Gamma^b\xi_2\right| + \sum_{b\leqslant a}\left|\Gamma^b\xi_1\right|\Bigg)\sum_{b\leqslant a}\left|\Gamma^b\xi_2\right|. \tag{6.12}$$

Of course, from (6.6), (6.9), (6.10) and (6.12), we obtain at once the following lemma:

Lemma 6.1. *One can find $\varepsilon_0, C > 0$ (both independent of T) such that the following holds: if $\varepsilon \leqslant \varepsilon_0$ and (θ_2, w_2, z_2) is a C^∞ solution of (3.12)–(3.17) when $x \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, with $E_\mu^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$, we have, if $|x| \geqslant r_1(t)$, $|a| \leqslant \mu$:*

(1) $|\Gamma^a \partial V_2(x, \frac{t}{\bar{c}})| + |\partial\Gamma^a V_2(x, \frac{t}{\bar{c}})| \leqslant C\sum_{b\leqslant a}|\Gamma^b\xi_2(x,t)|$,
(2) $|\Gamma^a\xi_2(x,t)| \leqslant C\sum_{b\leqslant a}|\partial\Gamma^b V_2(x, \frac{t}{\bar{c}})|$.

Lemma 6.2. *One can find $\varepsilon_0, C > 0$ (both independent of T) such that the following holds: if $\varepsilon \leqslant \varepsilon_0$ and (θ_2, w_2, z_2) is a C^∞ solution of (3.12)–(3.17) when $x \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, with $E_\mu^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$, we have if $|a| \leqslant \mu$,*

(1) $\sup_{r\geqslant r_1(t)}(r|\partial\Gamma^a V_2(x, \frac{t}{\bar{c}})|) \leqslant CE_{|a|+1}^{1/2}(t)$,
(2) $\sup_{r\geqslant r_1(t)}(r^{1/2}|\Gamma^a V_2(x, \frac{t}{\bar{c}})|) \leqslant CE_{|a|}^{1/2}(t)$.

Proof. (1) follows at once from the proof of (3.15b) of [16] and from Lemma 6.1(1). To prove (2) we are going to adapt the proof of Lemma 3.3(1) of [16]. Put $\mathcal{D}_R = \{y \in \mathbb{R}^3,\ |y| \geqslant R\}$, $\mathcal{D}_{R_1,R_2} = \{y \in \mathbb{R}^3,\ R_1 \leqslant |y| \leqslant R_2\}$. We are going to show that

$$\begin{aligned}&\text{for some } C>0,\ \text{all } R>0 \text{ and all } v\in C^\infty\big(\overline{\mathcal{D}}_R,\mathbb{R}\big)\\ &\quad\text{vanishing outside some compact subset of } \overline{\mathcal{D}}_R,\\ &\quad r^{1/2}\big|v(x)\big|\leqslant C\sum_{|b|\leqslant 1}\big\|\nabla\Omega^b v\big\|_{L^2(\mathcal{D}_R)} \text{ if } |x|\geqslant R,\\ &\quad\text{where } \Omega^b=\Omega_1^{b_1}\Omega_2^{b_2}\Omega_3^{b_3} \text{ if } b=(b_1,b_2,b_3)\in\mathbb{N}^3.\end{aligned}\tag{6.13}$$

Combining (6.13) (with $R=r_1(t)$ and $v(x)=\Gamma^a V_2(x,\frac{t}{c})$) and Lemma 6.1(1) immediately gives (2).

If $R=0$, (6.13) becomes just (3.15a) of [16]. It is enough to prove (6.13) for $R=1$ because the general case follows by considering the function $x\mapsto v(Rx)$ for $r\geqslant 1$. Now (6.13) for $R=1$ can be proved by a straightforward repetition of the argument of the proof of (3.15a) of [16] if we know that the following holds:

$$\begin{aligned}&\text{for some } C>0 \text{ and all } v\in C^\infty\big(\overline{\mathcal{D}}_1,\mathbb{R}\big)\\ &\text{vanishing outside some compact subset of } \overline{\mathcal{D}}_1,\ \|v\|_{L^6(\mathcal{D}_1)}\leqslant C\|\nabla v\|_{L^2(\mathcal{D}_1)}.\end{aligned}\tag{6.14}$$

In order to obtain Lemma 6.2(2) by the argument described above, it is important that ∇v, and not v itself, appears in the L^2 norm in the right-hand side of the inequality in (6.14). (6.14) must be well known but we are going to prove it since we do not know any reference. Let $\mathcal{E}$ be a standard Lions' extension operator defined by $\mathcal{E}v(x)=v(x)$ if $r\geqslant 1$; $\mathcal{E}v(x)=\chi(r)\sum_{1\leqslant j\leqslant 2}c_j v((1+\frac{1-r}{j})\frac{x}{r})$ if $0<r\leqslant 1$, where $\chi\in C^\infty(\mathbb{R},\mathbb{R})$ satisfies $\chi(r)=0$ if $r\leqslant 1/3$, $\chi(r)=1$ if $r\geqslant 2/3$, and $c_1=-3$, $c_2=4$; $\mathcal{E}v(0)=0$. By a well-known Sobolev estimate, $\|\mathcal{E}v\|_{L^6(\mathbb{R}^3)}\leqslant C\|\nabla\mathcal{E}v\|$, hence

$$\|\mathcal{E}v\|_{L^6(\mathbb{R}^3)}\leqslant C\big(\|\nabla v\|_{L^2(\mathcal{D}_1)}+\|v\|_{L^2(\mathcal{D}_{7/6,5/3})}\big).\tag{6.15}$$

Now, if $\omega\in S^2$, we have:

$$\int_{7/6}^{5/3} v^2(s\omega)s^2\,\mathrm{d}s\leqslant C\int_{7/6}^{5/3} v^2(s\omega)\,\mathrm{d}s.\tag{6.16}$$

On the other hand, integrating the identity $sv(s\omega)\partial_s(v(s\omega))=\partial_s(\frac{s}{2}v^2(s\omega))-\frac{1}{2}v^2(s\omega)$ with respect to s over $(7/6,+\infty)$ easily yields the inequality:

$$\int_{7/6}^{+\infty} v^2(s\omega)\,\mathrm{d}s\leqslant C\int_{7/6}^{+\infty} s^2\big(\partial_s\big(v(s\omega)\big)\big)^2\,\mathrm{d}s.\tag{6.17}$$

(6.16) and (6.17) imply that $\|v\|_{L^2(\mathcal{D}_{7/6,5/3})}\leqslant C\|\nabla v\|_{L^2(\mathcal{D}_{7/6})}$, so (6.14) now follows from (6.15). Hence (6.13) is proved. The proof of Lemma 6.2 is complete. □

Henceforth, we shall set if $f_1(x,t)$, $f_2(x,t)$ are $\mathbb{R}$-valued functions: $|f_1(t)f_2(\frac{t}{c})|_{+,t}=|\varphi(t)|_+$, $\|f_1(t)f_2(\frac{t}{c})\|_{+,t}=\|\varphi(t)\|_+$, where $\varphi(x,t)=f_1(x,t)f_2(x,\frac{t}{c})$.

Useful estimates are collected in the following lemma:

Lemma 6.3. *One can find $\varepsilon_0, C>0$ (both independent of T, with C independent of $\delta>0$) such that the following holds: if $\varepsilon\leqslant\varepsilon_0$ and (θ_2,w_2,z_2) is a C^∞ solution of* (3.12)–(3.17) *when $x\in\mathbb{R}^3$, $0\leqslant t\leqslant T$, with $E_\mu^{1/2}(t)\leqslant\varepsilon^2\psi(\varepsilon)$, we have if $\varepsilon\leqslant\varepsilon_0$, $|a|\leqslant\mu$, $\kappa(a)=0$ if $|a|\leqslant\mu-2$ and $\kappa(a)=\delta$ if $|a|=\mu-1,\mu$:*

(1) $|\Gamma^a(F_1(\xi))(t)|_+\leqslant C\frac{\varepsilon^2}{\langle t\rangle^{2-\kappa(a)}}$,
(2) $\|\Gamma^a(F_2(\xi_2,2\xi_1+\xi_2))(t)\|_+\leqslant C\frac{\varepsilon^3\psi(\varepsilon)}{\langle t\rangle}$,
(3) $\|\sigma_-(t)\Gamma^b\partial_t\theta_2(t)\|\leqslant C\varepsilon^2\psi(\varepsilon)$ *if* $|b|\leqslant\mu-1$,
(4) $\|\sigma_-(t)\Gamma^b\partial_t(F_2(\xi_2,2\xi_1+\xi_2))(t)\|_+\leqslant C\frac{\varepsilon^3\psi(\varepsilon)}{\langle t\rangle}$ *if* $|b|\leqslant\mu-1$,
(5) $\|\sigma_-(t)(\partial_i\partial_j\Gamma^b V_2)(\frac{t}{c})\|_{+,t}\leqslant C\varepsilon^2\psi(\varepsilon)$ *if* $0\leqslant i,j\leqslant 3$ *and* $|b|\leqslant\mu-1$.

Proof. We have:

$$\left|\Gamma^a\big(F_1(\xi)\big)(t)\right|_+ \leqslant C\Bigg(\sum_{\substack{|b|\leqslant[|a|/2]\\ b\leqslant a}}\left|\Gamma^b\xi(t)\right|_+\Bigg)\sum_{b\leqslant a}\left|\Gamma^b\xi(t)\right|_+,$$

so using Proposition 3.2(1), Lemma 4.1(2), and Theorem 3.2 with $m=\mu+2$, we obtain (1).

Using (6.12), Proposition 3.2(1) and Lemma 4.1(2), we can readily check (2).

To prove (3), notice that it follows from (4.1) that $\partial_t\Gamma^d\theta_2=-\nabla\cdot\Gamma^d w_2+h_0^d$, so using (6.10) and (B.1), (B.2) of Appendix B with $n=|\mu|$, we obtain (3).

Let us show (4). $\Gamma^b\partial_t(F_2(\xi_2,2\xi_1+\xi_2))$ is a linear combination of terms of the form $\tau_1=F_2(\Gamma^{b'}\partial_t^{l'}\xi_2,\Gamma^{b''}\partial_t^{l''}\xi_1)$ and $\tau_2=F_2(\Gamma^{b'}\xi_2,\Gamma^{b''}\partial_t\xi_2)$, where $b'+b''=b$, $l'+l''=1$. Assume now that $|b|\leqslant\mu-1$. If $l'=1$ and $l''=0$, we have:

$$\left\|\sigma_-(t)\tau_1(t)\right\|_+\leqslant\frac{1}{2}\left\|\sigma_-(t)\Gamma^{b'}\partial_t\xi_2(t)\right\|_+\left|\Gamma^{b''}\xi_1(t)\right|_+,$$

which can be bounded above by $C\varepsilon^2\psi(\varepsilon)\frac{\varepsilon}{\langle t\rangle}$ if we use (3), (6.10), Propositions 4.1 and 3.2(1). Notice that

$$\left|\Gamma^{b'}\xi_2(t)\right|_+\leqslant C\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle},\tag{6.18}$$

by Lemmas 6.1(2) and 6.2(1). If $l'=0$ and $l''=1$, we write:

$$\left\|\sigma_-(t)\tau_1(t)\right\|_+\leqslant\frac{1}{2}\left|\Gamma^{b'}\xi_2(t)\right|_+\left\|\sigma_-(t)\Gamma^{b''}\partial_t\xi_1(t)\right\|_+\leqslant C\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle}\varepsilon,$$

by (6.18) and Theorem A.1(2) (or Proposition 3.2(1)). Finally,

$$\left\|\sigma_-(t)\tau_2(t)\right\|_+\leqslant\frac{1}{2}\left|\Gamma^{b'}\xi_2(t)\right|_+\left\|\sigma_-(t)\Gamma^{b''}\partial_t\xi_2(t)\right\|_+,$$

which can be bounded above by $C\frac{\varepsilon^2\psi(\varepsilon)}{\langle t\rangle}\varepsilon^2\psi(\varepsilon)$ if we make use of (6.18), (3), (6.10) and Proposition 4.1. It is now easy to obtain (4).

Let us prove (5). We have by (6.11): $\partial_i\partial_j\Gamma^bV_2=\partial_i(\sum_{d\leqslant b}\hat{c}_{bd}\Gamma^d\partial_jV_2)$, so if $j>0$, (5) follows at once from (6.9), Lemma 4.1(1) (if $i>0$), and Proposition 4.1. If $j=0<i$, we exchange i and j and see by the preceding argument that (5) still holds. If now $i=j=0$, we apply (6.11) twice and find that we have for some $\tilde{c}_{bd}\in\mathbb{R}$: $\partial_t^2\Gamma^bV_2=\sum_{d\leqslant b}\tilde{c}_{bd}\Gamma^d\partial_t^2V_2$. But then (5) follows at once from (6.6), (3), (4). The proof of Lemma 6.3 is complete. □

Now put $\widehat{\Gamma}_j=\Gamma_j$ if $1\leqslant j\leqslant 5$, $\widehat{\Gamma}_{5+j}=\Omega_j$ if $1\leqslant j\leqslant 3$, $\widehat{\Gamma}^d=\widehat{\Gamma}_1^{d_1}\ldots\widehat{\Gamma}_8^{d_8}$ if $d=(d_1,\ldots,d_8)\in\mathbb{N}^8$. We shall need the following result, which is proved in [17].

Lemma 6.4. *Assume that $f^{ijk}\in\mathbb{R}$, $0\leqslant i,j,k\leqslant 3$, with $f^{ijk}=f^{ikj}$, and that $\sum_{0\leqslant i,j,k\leqslant 3}f^{ijk}p_ip_jp_k=0$ if $p=(p_0,\ldots,p_3)\in\mathbb{R}^4$ and $p_0^2=\bar{c}^2\sum_{1\leqslant j\leqslant 3}p_j^2$. Then one can find $C>0$ such that for all $f,g\in C^\infty(\{(x,t)\in\mathbb{R}^3\times[0,T],r\geqslant\frac{\bar{c}}{2}t+M+1\})$, we have, if $r\geqslant\frac{\bar{c}}{2}t+M+1$:*

$$\Bigg|\sum_{0\leqslant i,j,k\leqslant 3}f^{ijk}\partial_if\partial_{jk}^2g\Bigg|\leqslant\frac{C}{\sigma_+}\Bigg(\big|\widehat{\Gamma}f\big|\big|\partial^2g\big|+|\partial f|\sum_{0\leqslant h\leqslant 1}\big|\partial\widehat{\Gamma}^hg\big|+\bar{\sigma}_-|\partial f|\big|\partial^2g\big|\Bigg),$$

where $\partial^l\widehat{\Gamma}^n\varphi=\{\partial^\alpha\widehat{\Gamma}^\beta\varphi,\ |\alpha|=l,\ |\beta|=n\}$ if $l,n\in\mathbb{N}$, and $\bar{\sigma}_-(x,t)=\langle\bar{c}t-|x|\rangle$.

Observe that for some $c_{jpq},c_p\in\mathbb{R}$, $[\Omega_j,\Gamma^p]=\sum_{|q|=|p|}c_{jpq}\Gamma^q$, $[X,\Gamma^p]=c_p\Gamma^p$ if $p\in\mathbb{N}^5$. Hence from Lemma 6.4 we immediately obtain the following result.

Corollary 6.1. *Fix $p,q\in\mathbb{N}^5$. Under the assumptions of Lemma 6.4, one can find $C>0$ such that, for all f,g of the form $f=\Gamma^p\bar{f}$, $g=\Gamma^q\bar{g}$, with $\bar{f},\bar{g}$ smooth and radial, we have:*

$$\Big|\sum_{0\leqslant i,j,k\leqslant 3} f^{ijk}\partial_i f\partial_{jk}^2 g\Big| \leqslant \frac{C}{\sigma_+}\bigg(\sum_{\max(1,|p|)\leqslant|\tilde p|\leqslant|p|+1}\big|\Gamma^{\tilde p}\bar f\big|\big|\partial^2\Gamma^q\bar g\big| + \big|\partial\Gamma^p\bar f\big|\sum_{|q|\leqslant|\tilde q|\leqslant|q|+1}\big|\partial\Gamma^{\tilde q}\bar g\big| + \bar\sigma_-\big|\partial\Gamma^p\bar f\big|\big|\partial^2\Gamma^q\bar g\big|\bigg).$$

Corollary 6.1 will be applied repeatedly to obtain the decay estimates contained in the next lemma; these estimates will play a key role in the proof of Proposition 6.2.

Lemma 6.5. *One can find $\varepsilon_0, C>0$ (both independent of T, with C independent of $\delta>0$) such that the following holds: if $\varepsilon\leqslant\varepsilon_0$ and (θ_2,w_2,z_2) is a C^∞ solution of* (3.12)–(3.17) *when $x\in\mathbb{R}^3$, $0\leqslant t\leqslant T$, with $E_\mu^{1/2}(t)\leqslant\varepsilon^2\psi(\varepsilon)$, we have if $\ell\in\{1,2,3\}$ and $|a|\leqslant\mu$:*

(1) $\|(\Gamma^b\partial_t V\partial_l\Gamma^{a-b}\partial_l V_2-\Gamma^b\partial_l V\partial_l\Gamma^{a-b}\partial_t V_2)(\frac{t}{c})\|_{+,t}\leqslant C\frac{\varepsilon^3\psi(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$ *if* $0\neq b\leqslant a$,

(2) $\|(\Gamma^b\partial_t V_2\partial_l\Gamma^{a-b}\partial_l V_1-\Gamma^b\partial_l V_2\partial_l\Gamma^{a-b}\partial_t V_1)(\frac{t}{c})\|_{+,t}\leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$ *if* $b\leqslant a$,

(3) $\|(\Gamma^b\partial_t V\partial_l\Gamma^{a-b}\partial_t V_2-\bar c^2\sum_{1\leqslant j\leqslant 3}\Gamma^b\partial_j V\partial_j\Gamma^{a-b}\partial_l V_2)(\frac{t}{c})\|_{+,t}\leqslant C\frac{\varepsilon^3\psi(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$ *if* $0\neq b\leqslant a$,

(4) $\|(\Gamma^b\partial_t V_2\partial_l\Gamma^{a-b}\partial_t V_1-\bar c^2\sum_{1\leqslant j\leqslant 3}\Gamma^b\partial_j V_2\partial_j\Gamma^{a-b}\partial_l V_1)(\frac{t}{c})\|_{+,t}\leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$ *if* $b\leqslant a$,

(5) $\|(\Gamma^a\partial_t V_2\partial_l^2 V-\Gamma^a\partial_l V_2\partial_l\partial_t V)(\frac{t}{c})\|_{+,t}\leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$,

(6) $\|(\bar c^2\Gamma^a\partial_l V_2\triangle V-\Gamma^a\partial_t V_2\partial_l\partial_t V)(\frac{t}{c})\|_{+,t}\leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$.

Proof. (1) Set $B_{dn}(t)=\|(\partial_t\Gamma^d V\partial_l^2\Gamma^n V_2-\partial_l\Gamma^d V\partial_l\partial_t\Gamma^n V_2)(\frac{t}{c})\|_{+,t}$, where $d\leqslant b$ and $n\leqslant a-b$. Using Corollary 6.1 with $f^{ll0}=f^{l0l}=-1/2$, $f^{0ll}=1$, $f^{ijk}=0$ otherwise, $f=\Gamma^d V$, $g=\Gamma^n V_2$, we obtain that $B_{dn}(t)\leqslant\frac{C}{\langle t\rangle}\sum_{1\leqslant i\leqslant 3}\tau_{1i1}(t)\tau_{1i2}(t)$, where $\tau_{111}(t)=\sum_{|d|\leqslant|\tilde d|\leqslant|d|+1}|\Gamma^{\tilde d}V(\frac{t}{c})|_{+,t}$, $\tau_{112}(t)=\|\partial^2\Gamma^n V_2(\frac{t}{c})\|_{+,t}$, $\tau_{121}(t)=|\partial\Gamma^d V(\frac{t}{c})|_{+,t}$, $\tau_{122}(t)=\sum_{|n|\leqslant|\tilde n|\leqslant|n|+1}\|\partial\Gamma^{\tilde n}V_2(\frac{t}{c})\|_{+,t}$, $\tau_{131}=\tau_{121}$, $\tau_{132}(t)=\|\sigma_-(t)\partial^2\Gamma^n V_2(\frac{t}{c})\|_{+,t}$. Now $\tau_{111}(t)\leqslant C\frac{\varepsilon}{\langle t\rangle^{1/2-\delta}}$ by Theorem A.1(1), Lemma 6.2(2), and Theorem 3.2 with $m=\mu+1$; $\tau_{112}(t)+\tau_{132}(t)\leqslant C\varepsilon^2\psi(\varepsilon)$ by Lemma 6.3(5); $\tau_{121}(t)\leqslant C\frac{\varepsilon}{\langle t\rangle^{1-\delta}}$ by Theorem A.1(1), Lemma 6.2(1), and Theorem 3.2 with $m=\mu+1$; and $\tau_{122}(t)\leqslant C\varepsilon^2\psi(\varepsilon)$ by Lemma 6.1(1). Then (1) follows thanks to (6.10).

(2) Set $\widetilde B_{dn}(t)=\|(\partial_t\Gamma^d V_2\partial_l^2\Gamma^n V_1-\partial_l\Gamma^d V_2\partial_l\partial_t\Gamma^n V_1)(\frac{t}{c})\|_{+,t}$, where $d\leqslant b$ and $n\leqslant a-b$ ($\widetilde B_{dn}$ is obtained by replacing V,V_2 by V_2,V_1 in B_{dn}). Applying Corollary 6.1 with the same f^{ijk} as in the proof of (1), $f=\Gamma^d V_2$, $g=\Gamma^n V_1$, we obtain that $\widetilde B_{dn}(t)\leqslant\frac{C}{\langle t\rangle}\sum_{1\leqslant i\leqslant 3}\tau_{2i1}(t)\tau_{2i2}(t)$, where $\tau_{211}(t)=\sum_{|d|\leqslant|\tilde d|\leqslant|d|+1}|\Gamma^{\tilde d}V_2(\frac{t}{c})|_{+,t}$, $\tau_{212}(t)=\|\partial^2\Gamma^n V_1(\frac{t}{c})\|_{+,t}$, $\tau_{221}(t)=|\partial\Gamma^d V_2(\frac{t}{c})|_{+,t}$, $\tau_{222}(t)=\sum_{|n|\leqslant|\tilde n|\leqslant|n|+1}\|\partial\Gamma^{\tilde n}V_1(\frac{t}{c})\|_{+,t}$, $\tau_{231}=\tau_{221}$, $\tau_{232}(t)=\|\sigma_-(t)\partial^2\Gamma^n V_1(\frac{t}{c})\|_{+,t}$. Now $\tau_{211}(t)\leqslant C\frac{\varepsilon^2}{\langle t\rangle^{1/2-\delta}}$ by Lemma 6.2(2) and Theorem 3.2 with $m=\mu+1$; $\tau_{221}(t)\leqslant C\frac{\varepsilon^2}{\langle t\rangle^{1-\delta}}$ by Lemma 6.2(1) and Theorem 3.2 with $m=\mu+1$; $\sum_{1\leqslant i\leqslant 3}\tau_{2i2}(t)\leqslant C\varepsilon$ by Theorem A.1(2). Then (2) follows thanks to (6.10).

(3) Put $P_{dn}(t)=\|(\partial_t\Gamma^d V\partial_l\partial_t\Gamma^n V_2-\bar c^2\sum_{1\leqslant j\leqslant 3}\partial_j\Gamma^d V\partial_{jl}^2\Gamma^n V_2)(\frac{t}{c})\|_{+,t}$, where $d\leqslant b$ and $n\leqslant a-b$. Using Corollary 6.1 with $f^{jjl}=f^{jlj}=-\frac{\bar c^2}{2}$ if $j>0$, $f^{00l}=f^{0l0}=\frac12$, $f^{ijk}=0$ otherwise, $f=\Gamma^d V$, $g=\Gamma^n V_2$, we obtain that $P_{dn}(t)\leqslant\frac{C}{\langle t\rangle}\sum_{1\leqslant i\leqslant 3}\tau_{1i1}(t)\tau_{1i2}(t)$, with τ_{1ij} as in the proof of (1). Hence (3) follows at once from the estimates obtained in the proof of (1) if we use (6.10).

(4) Put $\widetilde P_{dn}(t)=\|(\partial_t\Gamma^d V_2\partial_l\partial_t\Gamma^n V_1-\bar c^2\sum_{1\leqslant j\leqslant 3}\partial_j\Gamma^d V_2\partial_{jl}^2\Gamma^n V_1)(\frac{t}{c})\|_{+,t}$, where $d\leqslant b$ and $n\leqslant a-b$. ($\widetilde P_{dn}$ is obtained by replacing V,V_2 by V_2,V_1 in P_{dn}.) Applying Corollary 6.1 with the same f^{ijk} as in (3), $f=\Gamma^d V_2$, $g=\Gamma^n V_1$, we obtain that $\widetilde P_{dn}(t)\leqslant\frac{C}{\langle t\rangle}\sum_{1\leqslant i\leqslant 3}\tau_{2i1}(t)\tau_{2i2}(t)$, with τ_{2ij} as in the proof of (2). Hence (4) follows at once from the estimates obtained in the proof of (2) if we use (6.10).

(5) Put $B_d(t)=\|(\partial_t\Gamma^d V_2\partial_l^2 V-\partial_l\Gamma^d V_2\partial_l\partial_t V)(\frac{t}{c})\|_{+,t}$, where $d\leqslant a$. Using Corollary 6.1 with $f^{0ll}=1$, $f^{ll0}=f^{l0l}=-\frac12$, $f^{ijk}=0$ otherwise, $f=\Gamma^d V_2$, $g=V$, we obtain that $B_d(t)\leqslant\frac{C}{\langle t\rangle}\sum_{1\leqslant i\leqslant 3}\tau_{3i1}(t)\tau_{3i2}(t)$, with $\tau_{311}=\tau_{211}$, $\tau_{312}(t)=\|\partial^2 V(\frac{t}{c})\|_{+,t}$, $\tau_{321}=\tau_{331}=\tau_{221}$, $\tau_{322}(t)=\sum_{|n|\leqslant 1}\|\partial\Gamma^n V(\frac{t}{c})\|_{+,t}$, $\tau_{332}(t)=\|\sigma_-(t)\partial^2 V(\frac{t}{c})\|_{+,t}$. τ_{211}

and τ_{221} have been estimated in the proof of (2), and $\sum_{1\leqslant i\leqslant 3}\tau_{3i2}(t)\leqslant C\varepsilon$ by Theorem A.1(2) and Lemmas 6.1(1) and 6.3(5). Hence $B_d(t)\leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$, from which (5) follows thanks to (6.10).

(6) Put $\widetilde{B}_d(t)=\|(\bar{c}^2\partial_l\Gamma^d V_2\Delta V-\partial_t\Gamma^d V_2\partial_l\partial_t V)(\frac{t}{\bar{c}})\|_{+,t}$, where $d\leqslant a$. Using Corollary 6.1 with $f^{ljj}=\bar{c}^2$ if $j>0$, $f^{0l0}=f^{00l}=-\frac{1}{2}$, $f^{ijk}=0$ otherwise, $f=\Gamma^d V_2$, $g=V$, we obtain that $\widetilde{B}_d(t)\leqslant\frac{C}{\langle t\rangle}\sum_{1\leqslant i\leqslant 3}\tau_{3i1}(t)\tau_{3i2}(t)$, with τ_{3ij} as in the proof of (5). Hence (6) follows from the estimates given in the proof of (5) and from (6.10). The proof of Lemma 6.5 is complete. $\square$

We can now prove Proposition 6.2.

Proof of Proposition 6.2(1). If $b\leqslant a$, set $F_{ij}^{ab}=\Gamma^b\theta_i\nabla\cdot\Gamma^{a-b}w_j-\Gamma^b w_i\cdot\nabla\Gamma^{a-b}\theta_j$, $G_{ij}^{ab}=\Gamma^b\theta_i\nabla\Gamma^{a-b}\theta_j-\Gamma^b w_i\cdot\nabla\Gamma^{a-b}w_j$. Using Proposition 4.3, we find that $\langle F^a,\zeta^a\rangle_+=\langle\hat{h}_0^a,\Gamma^a\theta_2\rangle_++\langle\frac{\hat{h}^a}{1+z},\Gamma^a w_2\rangle_+$ and

$$\hat{h}_0^a=\sum_{\substack{(i,j)\neq(1,1)\\ b\leqslant a,\ |b|\geqslant\beta_{ij}}}\binom{a}{b}F_{ij}^{ab},\qquad \hat{h}^a=\sum_{\substack{(i,j)\neq(1,1)\\ b\leqslant a,\ |b|\geqslant\beta_{ij}}}\binom{a}{b}G_{ij}^{ab}\quad\text{if } |x|\geqslant M+1,$$

where $\beta_{12}=\beta_{22}=1$, $\beta_{21}=0$. It is clear that Proposition 6.2(1) immediately follows from the following result:

Lemma 6.6. *One can find $\varepsilon_0, C>0$ (both independent of T, with C independent of $\delta>0$) such that the following holds: if $\varepsilon\leqslant\varepsilon_0$ and (θ_2,w_2,z_2) is a C^∞ solution of (3.12)–(3.17) when $x\in\mathbb{R}^3$, $0\leqslant t\leqslant T$, with $E_\mu^{1/2}(t)\leqslant\varepsilon^2\psi(\varepsilon)$, we have if $|a|\leqslant\mu$:*

(1) $\|(F_{12}^{ab}+F_{22}^{ab})(t)\|_+\leqslant C\frac{\varepsilon^3\psi(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$ *if* $0\neq b\leqslant a$,
(2) $\|F_{21}^{ab}(t)\|_+\leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$ *if* $b\leqslant a$,
(3) $\|(G_{12}^{ab}+G_{22}^{ab})(t)\|_+\leqslant C\frac{\varepsilon^3\psi(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$ *if* $0\neq b\leqslant a$,
(4) $\|G_{21}^{ab}(t)\|_+\leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$ *if* $b\leqslant a$.

Proof of Lemma 6.6(1). Assume that $0\neq b\leqslant a$. Applying Γ^b to (6.5), (6.8), $\nabla\cdot\Gamma^{a-b}$ to (6.9) and $\nabla\Gamma^{a-b}$ to (6.6), we readily arrive at

$$\big\|\big(F_{12}^{ab}+F_{22}^{ab}\big)(t)\big\|_+\leqslant C\sum_{1\leqslant k\leqslant 3}K_k(t),$$

where

$$K_1(t)=\sum_{1\leqslant l\leqslant 3}\left\|\big(\Gamma^b\partial_t V\partial_l\Gamma^{a-b}\partial_l V_2-\Gamma^b\partial_l V\partial_l\Gamma^{a-b}\partial_t V_2\big)\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$K_2(t)=\sum_{1\leqslant l\leqslant 3}\left\|\Gamma^b\big(F_1(\xi)\big)(t)\partial_l\Gamma^{a-b}\partial_l V_2\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$K_3(t)=\sum_{1\leqslant l\leqslant 3}\left\|\Gamma^b\partial_l V\left(\frac{t}{\bar{c}}\right)\partial_l\Gamma^{a-b}\big(F_2(\xi_2,2\xi_1+\xi_2)\big)(t)\right\|_{+,t}.$$

By Lemma 6.5(1), $K_1(t)\leqslant C\frac{\varepsilon^3\psi(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$. By Lemma 6.1(1), $\|\partial_l\Gamma^{a-b}\partial_l V_2(\frac{t}{\bar{c}})\|_{+,t}\leqslant CE_\mu^{1/2}(t)\leqslant C\varepsilon^2\psi(\varepsilon)$, so using Lemma 6.3(1) we find that $K_2(t)\leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^{2-\delta}}$. Now $|\Gamma^b\partial_l V(\frac{t}{\bar{c}})|_{+,t}\leqslant C\frac{\varepsilon}{\langle t\rangle^{1-\delta}}$ by Theorem A.1(1), (6.10), Lemma 6.2(1), and Theorem 3.2(1) with $m=\mu+1$. So using Lemma 6.3(2), we conclude that $K_3(t)\leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^{2-\delta}}$. Collecting estimates, we obtain (1).

(2) Assume that $b \leqslant a$. Applying Γ^b to (6.6), (6.9), $\nabla \cdot \Gamma^{a-b}$ to (6.7) and $\nabla \Gamma^{a-b}$ to (6.4), we arrive at

$$\|F_{21}^{ab}(t)\|_+ \leqslant C \sum_{1\leqslant k\leqslant 3} \widetilde{K}_k(t),$$

where

$$\widetilde{K}_1(t) = \sum_{1\leqslant l\leqslant 3} \left\| \left(\Gamma^b \partial_t V_2 \partial_l \Gamma^{a-b} \partial_l V_1 - \Gamma^b \partial_l V_2 \partial_l \Gamma^{a-b} \partial_t V_1\right)\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$\widetilde{K}_2(t) = \sum_{1\leqslant l\leqslant 3} \left\| \Gamma^b\big(F_2(\xi_2, 2\xi_1+\xi_2)\big)(t)\partial_l \Gamma^{a-b}\partial_l V_1\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$\widetilde{K}_3(t) = \sum_{1\leqslant l\leqslant 3} \left\| \Gamma^b \partial_l V_2\left(\frac{t}{\bar{c}}\right)\partial_l \Gamma^{a-b}\big(F_1(\xi_1)\big)(t)\right\|_{+,t}.$$

By Lemma 6.5(2), $\widetilde{K}_1(t) \leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$. By Lemma 6.3(2) and Theorem A.1(1), $\widetilde{K}_2(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^2}$. Using Lemma 6.1(1) and Proposition 3.2(1), we see that $\widetilde{K}_3(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^2}$. Collecting estimates, we obtain (2).

(3) Assume that $0 \neq b \leqslant a$. Applying Γ^b to (6.5), (6.8) and $\nabla \Gamma^{a-b}$ to (6.6), (6.9), we see that

$$\left\|\big(G_{12}^{ab} + G_{22}^{ab}\big)(t)\right\|_+ \leqslant C \sum_{1\leqslant k\leqslant 4} R_k(t),$$

where

$$R_1(t) = \sum_{1\leqslant l\leqslant 3} \left\| \left(\Gamma^b \partial_t V \partial_l \Gamma^{a-b} \partial_t V_2 - \bar{c}^2 \sum_{1\leqslant j\leqslant 3} \Gamma^b \partial_j V \partial_j \Gamma^{a-b} \partial_l V_2\right)\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$R_2(t) = \sum_{1\leqslant l\leqslant 3} \left\| \Gamma^b \partial_t V\left(\frac{t}{\bar{c}}\right)\partial_l \Gamma^{a-b}\big(F_2(\xi_2, 2\xi_1+\xi_2)\big)(t)\right\|_{+,t},$$

$$R_3(t) = \sum_{1\leqslant l\leqslant 3} \left\| \Gamma^b\big(F_1(\xi)\big)(t)\partial_l \Gamma^{a-b}\partial_t V_2\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$R_4(t) = \sum_{1\leqslant l\leqslant 3} \left\| \Gamma^b\big(F_1(\xi)\big)(t)\partial_l \Gamma^{a-b}\big(F_2(\xi_2, 2\xi_1+\xi_2)\big)(t)\right\|_+.$$

By Lemma 6.5(3), $R_1(t) \leqslant C\frac{\varepsilon^3\psi(\varepsilon)}{\langle t\rangle^{3/2-\delta}}$. Arguing as we did for K_3 (resp. K_2) (in (1)), we find that $R_2(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^{2-\delta}}$ (resp. $R_3(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^{2-\delta}}$). Finally, by Lemma 6.3(1), (2), we find that $R_4(t) \leqslant C\frac{\varepsilon^5\psi(\varepsilon)}{\langle t\rangle^{3-\delta}}$. Collecting estimates, we obtain (3).

(4) Assume that $b \leqslant a$. Applying Γ^b to (6.6), (6.9) and $\nabla \Gamma^{a-b}$ to (6.4), (6.7), we see that

$$\|G_{21}^{ab}(t)\|_+ \leqslant C \sum_{1\leqslant k\leqslant 4} \widetilde{R}_k(t),$$

where

$$\widetilde{R}_1(t) = \sum_{1\leqslant l\leqslant 3} \left\| \left(\Gamma^b \partial_t V_2 \partial_l \Gamma^{a-b} \partial_t V_1 - \bar{c}^2 \sum_{1\leqslant j\leqslant 3} \Gamma^b \partial_j V_2 \partial_j \Gamma^{a-b} \partial_l V_1\right)\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$\widetilde{R}_2(t) = \sum_{1\leqslant l\leqslant 3} \left\| \Gamma^b \partial_t V_2\left(\frac{t}{\bar{c}}\right)\partial_l \Gamma^{a-b}\big(F_1(\xi_1)\big)(t)\right\|_{+,t},$$

$$\widetilde{R}_3(t) = \sum_{1\leqslant l\leqslant 3} \left\| \Gamma^b\big(F_2(\xi_2, 2\xi_1+\xi_2)\big)(t)\partial_l \Gamma^{a-b}\partial_t V_1\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$\widetilde{R}_4(t) = \sum_{1\leqslant l\leqslant 3} \left\| \Gamma^b\big(F_2(\xi_2, 2\xi_1+\xi_2)\big)(t)\partial_l \Gamma^{a-b}\big(F_1(\xi_1)\big)(t)\right\|_+.$$

By Lemma 6.5(4), $\widetilde{R}_1(t) \leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$. Arguing as we did for $\widetilde{K}_3$ (resp. $\widetilde{K}_2$) (in the proof of (2)), we find that $\widetilde{R}_2(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^2}$ (resp. $\widetilde{R}_3(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^2}$). Finally, by Lemma 6.3(2) and Proposition 3.2(1), we find that $\widetilde{R}_4(t) \leqslant C\frac{\varepsilon^5\psi(\varepsilon)}{\langle t\rangle^3}$. Collecting estimates, we obtain (4).

The proof of Lemma 6.6 is complete, and so is the proof of Proposition 6.2(1). □

Proof of Proposition 6.2(2). Set $S_1^a = (\nabla\cdot w)\Gamma^a\theta_2 - \nabla\theta\cdot\Gamma^a w_2$, $S_2^a = (\nabla\cdot w)\Gamma^a w_2 - (\Gamma^a\theta_2)\nabla\theta$. We have:

$$\left\langle \sum_{0\leqslant j\leqslant 3}(\partial_j A_j)\zeta^a, \zeta^a\right\rangle_+ = \left\langle S_1^a, \Gamma^a\theta_2\right\rangle_+ + \left\langle S_2^a, \Gamma^a w_2\right\rangle_+.$$

So it is clear that Proposition 6.2(2) immediately follows from the next lemma.

Lemma 6.7. *One can find $\varepsilon_0, C > 0$ (both independent of T, with C independent of $\delta > 0$) such that the following holds: if $\varepsilon \leqslant \varepsilon_0$ and (θ_2, w_2, z_2) is a C^∞ solution of (3.12)–(3.17) when $x \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, with $E_\mu^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$, we have if $|a| \leqslant \mu$:*

(1) $\|S_1^a(t)\|_+ \leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$,
(2) $\|S_2^a(t)\|_+ \leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$.

Proof. (1) Applying Γ^a to (6.6), (6.9), $\nabla\cdot$ to (6.8) and ∇ to (6.5), we see that

$$\left\|S_1^a(t)\right\|_+ \leqslant C\sum_{1\leqslant k\leqslant 3} Z_k(t),$$

where

$$Z_1(t) = \sum_{1\leqslant l\leqslant 3}\left\|\left(\partial_l^2 V\Gamma^a\partial_t V_2 - \partial_l\partial_t V\Gamma^a\partial_l V_2\right)\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$Z_2(t) = \sum_{1\leqslant l\leqslant 3}\left\|\partial_l^2 V\left(\frac{t}{\bar{c}}\right)\Gamma^a\left(F_2(\xi_2, 2\xi_1+\xi_2)\right)(t)\right\|_{+,t},$$

$$Z_3(t) = \sum_{1\leqslant l\leqslant 3}\left\|\partial_l\left(F_1(\xi)\right)(t)\Gamma^a\partial_l V_2\left(\frac{t}{\bar{c}}\right)\right\|_{+,t}.$$

By Lemma 6.5(5), $Z_1(t) \leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$. Now $|\partial_l^2 V(\frac{t}{\bar{c}})|_{+,t} \leqslant C\frac{\varepsilon}{\langle t\rangle}$ thanks to Theorem A.1(1) and Lemma 6.2(1); hence, by Lemma 6.3(2), we obtain that $Z_2(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^2}$. Using Lemmas 6.3(1) and 6.1(1), we find that $Z_3(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^2}$. Collecting estimates, we obtain (1).

(2) Applying Γ^a to (6.9), (6.6), $\nabla\cdot$ to (6.8) and ∇ to (6.5), we see that

$$\left\|S_2^a(t)\right\|_+ \leqslant C\sum_{1\leqslant k\leqslant 4}\widetilde{Z}_k(t),$$

where

$$\widetilde{Z}_1(t) = \sum_{1\leqslant l\leqslant 3}\left\|\left(\bar{c}^2\Delta V\Gamma^a\partial_l V_2 - \Gamma^a\partial_t V_2\partial_l\partial_t V\right)\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$\widetilde{Z}_2(t) = \sum_{1\leqslant l\leqslant 3}\left\|\Gamma^a\partial_t V_2\left(\frac{t}{\bar{c}}\right)\partial_l\left(F_1(\xi)\right)(t)\right\|_{+,t},$$

$$\widetilde{Z}_3(t) = \sum_{1\leqslant l\leqslant 3}\left\|\Gamma^a\left(F_2(\xi_2, 2\xi_1+\xi_2)\right)(t)\partial_l\partial_t V\left(\frac{t}{\bar{c}}\right)\right\|_{+,t},$$

$$\widetilde{Z}_4(t) = \sum_{1\leqslant l\leqslant 3} \big\| \Gamma^a\big(F_2(\xi_2, 2\xi_1+\xi_2)\big)(t)\partial_l\big(F_1(\xi)\big)(t)\big\|_+.$$

By Lemma 6.5(6), $\widetilde{Z}_1(t) \leqslant C\frac{\varepsilon^3}{\langle t\rangle^{3/2-\delta}}$. Arguing as we did for Z_3 (resp. Z_2) (in (1)), we obtain that $\widetilde{Z}_2(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^2}$ (resp. $\widetilde{Z}_3(t) \leqslant C\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle^2}$). Using Lemma 6.3(2), (1) we find that $\widetilde{Z}_4(t) \leqslant C\frac{\varepsilon^5\psi(\varepsilon)}{\langle t\rangle^3}$. Collecting estimates, we obtain (2).

The proof of Lemma 6.6 is complete. Hence Proposition 6.2 is completely proved. □

We are now able to prove Theorem 3.3.

Proof of Theorem 3.3. Assume that ε is so small that $|z| \leqslant \frac{1}{2}$ if $x\in\mathbb{R}^3$, $0\leqslant t\leqslant T$. Integrating (4.19) when $|a|\leqslant\mu$ over $[0, T_1]$, where $0\leqslant T_1\leqslant T$, and using Propositions 6.1 and 6.2 with $\delta<\frac{1}{2}$, and (4.20), we obtain that

$$E_\mu(t) \leqslant C\varepsilon^4\psi(\varepsilon) \leqslant \left(\frac{1}{2}\varepsilon^2\psi(\varepsilon)\right)^2 \quad \text{if } \varepsilon\leqslant\varepsilon_0(\delta).$$

The proof of Theorem 3.3 is complete. □

Appendix A. Estimates for solutions of (3.3)–(3.5)

Theorem A.1. *One can find $\varepsilon_0, C_{a\alpha}, C_a > 0$ such that the following holds: if $\varepsilon\leqslant\varepsilon_0$, then* (3.3)–(3.5) *has a unique $C^\infty(\mathbb{R}^3\times[0,+\infty))$ solution V_1 which satisfies the estimates*

(1) $|\bar{\Lambda}^a\partial^\alpha V_1(x,t)| \leqslant C_{a\alpha}\varepsilon\langle\bar{c}t-|x|\rangle^{-|\alpha|-1}\langle\bar{c}t+|x|\rangle^{-1}$,
(2) $\|\bar{\Lambda}^a V_1(t)\| + \|\langle\bar{c}t-|\cdot|\rangle\bar{\Lambda}^a\partial V_1(t)\| \leqslant C_a\varepsilon$.

Proof. (3.3) can be written,

$$\big(\partial_t^2 - \bar{c}^2\Delta\big)V_1 = \sum_{0\leqslant i,j\leqslant 3} f^{ij}(\partial V_1)\partial_{ij}^2 V_1, \tag{A.1}$$

with $f^{ij}=f^{ji}\in C^\infty(\mathbb{R}^4)$ (actually the f^{ij} are polynomials), where (f^{ij}) satisfies the null condition of [7,2], namely $f^{ij}(p) = \sum_{0\leqslant k\leqslant 3} f^{ijk}p_k + \mathrm{O}(|p|^2)$ as $p=(p_0,p_1,p_2,p_3)\to 0$ in $\mathbb{R}^4$, where $f^{ijk}\in\mathbb{R}$ and $\sum_{0\leqslant i,j,k\leqslant 3} f^{ijk}q_iq_jq_k = 0$ for all $q=(q_0,q_1,q_2,q_3)\in\mathbb{R}^4$ which satisfy the condition $q_0^2 = \bar{c}^2\sum_{1\leqslant k\leqslant 3} q_k^2$. (Actually, in the case of (3.3), $\sum_{0\leqslant i,j,k\leqslant 3} f^{ijk}q_iq_jq_k = 0$ for all $q\in\mathbb{R}^4$, so more than the usual null condition is satisfied.) Furthermore,

$$\partial_t^j V_1(x,0) = f_j(x,\varepsilon) \quad \text{if } j=0,1, \tag{A.2}$$

where for some $\varepsilon_0, C_\alpha>0$, $|\partial_x^\alpha f_j|\leqslant C_\alpha\varepsilon$ if $\varepsilon\leqslant\varepsilon_0$, and $f_j(x,\varepsilon)=0$ if $|x|\geqslant M$ and $\varepsilon\leqslant\varepsilon_0$. The global existence of V_1 for ε small follows from [2,7]. To obtain the estimates, put $\bar{V}_1(x,t) = V_1(x,\frac{t}{\bar{c}})$ (this is by no means essential but enables us to use immediately the computations of [4]). Then from (A.1) it follows that

$$(\partial_t^2 - \Delta)\bar{V}_1 = \sum_{0\leqslant i,j\leqslant 3} \tilde{f}^{ij}\big(\partial\bar{V}_1\big)\partial_{ij}^2\bar{V}_1, \tag{A.3}$$

where the $\tilde{f}^{ij}$ are polynomials, $\tilde{f}^{ij}=\tilde{f}^{ji}$ and $(\tilde{f}^{ij})$ satisfies the null condition (once more for all $q\in\mathbb{R}^4$), now with $\bar{c}=1$. Proceeding as in Section 4 of [4], let us make a translation in time: fix $t_0>2M$ and solve (A.3) for $t>t_0$ with the initial conditions

$$\partial_t^j\bar{V}_1(x,t_0) = (\bar{c})^{-j}f_j(x,\varepsilon) \quad \text{if } j=0,1. \tag{A.4}$$

First we solve (A.3), (A.4) when $t\geqslant t_0$ and $t^2-|x|^2-t_0t\leqslant 0$. To continue the solution when $t^2-|x|^2-t_0t>0$, we use the conformal inversion $(y,s)=J(x,t)$ given by $y=\frac{x}{|x|^2-t^2}$, $s=\frac{t}{|x|^2-t^2}$. If $Z(y,s)=\frac{\bar{V}_1(J^{-1}(y,s))}{s^2-|y|^2}$, it is checked in formula (4.4) of [4] that

$$\left(D_0^2 - \sum_{1\leqslant j\leqslant 3} D_j^2\right)Z = \sum_{0\leqslant i,j\leqslant 3} g^{ij}(y,s,Z,DZ)D_{ij}^2 Z + g(y,s,Z,DZ), \tag{A.5}$$

where $D=(D_0,\dots,D_3)$ with $D_0=\partial/\partial s$, $D_j=\partial/\partial y_j$ if $j>0$, g^{ij} and $g\in C^\infty(\mathbb{R}^9)$ (actually, here, the g^{ij} and g are polynomials), $g^{ij}=g^{ji}$, and $g^{ij}(y,s,0,0)=\partial^\alpha_{Z,L}g(y,s,Z,L)|_{Z=L=0}\equiv 0$ if $|\alpha|\leqslant 1$. Furthermore

$$D_0^j Z\left(y,-\frac{1}{t_0}\right)=\tilde f_j(y,\varepsilon)\quad \text{if } j=0,1, \tag{A.6}$$

where, for some ε_0 and all $\varepsilon\leqslant\varepsilon_0$, $\tilde f_j(y,\varepsilon)=0$ if $|y|\geqslant R$ for some $R<\frac{1}{t_0}$ independent of ε, and $|\partial_y^\alpha \tilde f_j|\leqslant \widetilde C_\alpha\varepsilon$ for some $\widetilde C_\alpha>0$ independent of ε. Standard results show that (A.5), (A.6) has a $C^\infty(\mathbb{R}^3\times[-\frac{1}{t_0},0])$ solution Z if ε is small, with $|D^\alpha Z|\leqslant C_\alpha\varepsilon$ (see e.g. [4]). From this we obtain that

$$\left|\bar\Lambda^a V_1(x,t)\right|\leqslant C_a\varepsilon\left\langle \bar c t-|x|\right\rangle^{-1}\left\langle \bar c t+|x|\right\rangle^{-1}. \tag{A.7}$$

Since

$$\left|\partial^\alpha\bar\Lambda^b V_1(x,t)\right|\leqslant C_\alpha\left\langle\bar c t-|x|\right\rangle^{-|\alpha|}\sum_{|d|\leqslant|\alpha|}\left|\bar\Lambda^d\bar\Lambda^b V_1(x,t)\right|$$

(cf. [8]), (1), (2) follow easily from (A.7). The proof of Theorem A.1 is complete. □

Appendix B. Proof of Proposition 4.1

Using (4.3), (4.4), (4.2) (with $|a|\leqslant n-1$) we obtain that

$$Q_n(t)\leqslant C\left(E_n^{1/2}(t)+\sum_{|a|\leqslant n-1}t\left\|h_0^a(t)\right\|+\sum_{|a|\leqslant n-1}\left\|\left(t+|\cdot|\right)h^a(t)\right\|\right)$$

(cf. estimate (37) of [5]). Using Proposition 3.2(1), we find that

$$\left\|\sigma_+(t)\tau_j^a(t)\right\|\leqslant C\varepsilon E_n^{1/2}(t)\quad \text{if } j\in\{1,2,4,5,7,8,10,11\}. \tag{B.1}$$

On the other hand, we have the following estimates:

$$\begin{aligned}
&\left\|\sigma_+(t)\left(\Gamma^b\theta_2\nabla\Gamma^{a-b}\theta_2\right)(t)\right\|+\left\|\sigma_+(t)\left(\Gamma^b w_2\cdot\nabla\Gamma^{a-b}\theta_2\right)(t)\right\|\\
&\quad\leqslant C\left(E^{1/2}_{[(n+3)/2]}(t)Q_n(t)+E^{1/2}_{n-1}(t)Q_{[n/2]+2}(t)\right),\\
&\left\|\sigma_+(t)\left(\Gamma^b\theta_2\nabla\cdot\Gamma^{a-b}w_2\right)(t)\right\|\leqslant C\left(E^{1/2}_{[(n+3)/2]}(t)Q_n(t)+E^{1/2}_{n-1}(t)\widetilde Q_{[n/2]+2}(t)\right),\\
&\left\|\sigma_+(t)\left(\Gamma^b w_2\cdot\nabla\Gamma^{a-b}w_2\right)(t)\right\|\leqslant C\left(E^{1/2}_{[(n+3)/2]}(t)\widetilde Q_n(t)+E^{1/2}_{n-1}(t)\widetilde Q_{[n/2]+2}(t)\right).
\end{aligned} \tag{B.2}$$

These estimates are easily obtained by considering separately the cases $|b|\leqslant|a-b|$ and $|b|>|a-b|$, estimating higher derivatives in L^2 and lower derivatives in L^∞, and making use of Lemma 4.1. It follows from (B.2) that

$$\left\|\sigma_+(t)\tau_j^a(t)\right\|\leqslant C\left(E^{1/2}_{[(n+3)/2]}(t)\widetilde Q_n(t)+E^{1/2}_{n-1}(t)\widetilde Q_{[n/2]+2}(t)\right)\quad \text{if } j\in\{3,6,9,12\}. \tag{B.3}$$

To handle τ_{13}^a, put $\mathcal{L}_{ijb}(t)=\|\sigma_+(t)(\Gamma^b z_i\nabla\Gamma^{a-b}\theta_j)(t)\|$. Using Proposition 3.2(2), we find that $\mathcal{L}_{11b}(t)\leqslant C\frac{\varepsilon^2}{\langle t\rangle^3}$; and $\mathcal{L}_{12b}(t)\leqslant C\varepsilon Q_n(t)$, $\mathcal{L}_{21b}(t)\leqslant C\varepsilon E^{1/2}_{n-1}(t)$. As for $\mathcal{L}_{22b}(t)$, it can be estimated by $C|\sigma\Gamma^b z_2(t)|\|\sigma_-(t)\nabla\Gamma^{a-b}\theta_2(t)\|$ if $|b|\leqslant|a-b|$ and by $\|\Gamma^b z_2(t)\||\sigma_+(t)\nabla\Gamma^{a-b}\theta_2(t)|$ if $|b|>|a-b|$. So if we set

$$M_{j-1}(t)=\frac{\varepsilon^2}{\langle t\rangle^3}+\left(\varepsilon+Q_{[j/2]+2}(t)\right)E^{1/2}_{j-1}(t)+\left(\varepsilon+E^{1/2}_{[(j+3)/2]}(t)\right)Q_j(t),$$

we obtain with the help of Lemma 4.1(2), (3) that

$$\sum_{|\alpha|+k\leqslant n-1}\left\|\sigma_+(t)\partial^\alpha(X+1)^k(z\nabla\theta)(t)\right\|\leqslant CM_{n-1}(t). \tag{B.4}$$

Assume that $|\alpha|+k\leqslant n-1$. If $\alpha=\alpha'+\alpha''$, $k=k'+k''$, it follows from (B.4) and Proposition 3.2(1) that

$$\left\|\sigma_+(t)\partial^{\alpha'}X^{k'}\theta_1(t)\partial^{\alpha''}(X+1)^{k''}(z\nabla\theta)(t)\right\|\leqslant C\varepsilon\frac{M_{n-1}(t)}{\langle t\rangle}.$$

If $|\alpha'| + k' \leqslant |\alpha''| + k''$, we obtain:

$$\left\|\sigma_+(t)\partial^{\alpha'}X^{k'}\theta_2(t)\partial^{\alpha''}(X+1)^{k''}(z\nabla\theta)(t)\right\| \leqslant CE^{1/2}_{[(n+3)/2]}(t)M_{n-1}(t),$$

with the help of (B.4) and the Sobolev injection theorem. If $|\alpha'| + |k'| > |\alpha''| + k''$, we write

$$\left\|\sigma_+(t)\partial^{\alpha'}X^{k'}\theta_2(t)\partial^{\alpha''}(X+1)^{k''}(z\nabla\theta)(t)\right\| \leqslant CE^{1/2}_{n-1}(t)\left|\tilde{\sigma}_+(t)\partial^{\alpha''}(X+1)^{k''}(z\nabla\theta)(t)\right|,$$

where $\tilde{\sigma}_+(x,t) = (1+|x|^2+t^2)^{1/2}$. So making use of the Sobolev injection theorem and of (B.4), we then obtain that

$$\left\|\sigma_+(t)\partial^{\alpha'}X^{k'}\theta_2(t)\partial^{\alpha''}(X+1)^{k''}(z\nabla\theta)(t)\right\| \leqslant CE^{1/2}_{n-1}(t)M_{[n/2]+1}(t)$$

if $|\alpha'| + k' > |\alpha''| + k''$. Taking all this into account, we find that

$$\left\|\sigma_+(t)\tau^a_{13}(t)\right\| \leqslant C\big(1+E^{1/2}_{[(n+3)/2]}(t)\big)M_{n-1}(t) + CE^{1/2}_{n-1}(t)M_{[n/2]+1}(t). \tag{B.5}$$

Now, if $\eta > 0$ is small and $E^{1/2}_{[n/2]+2}(t) \leqslant \eta$, we can repeat the proof of Proposition 3 of [5] with Proposition 2 of [5] replaced by Proposition 3.2 of the present paper and the constant B of [5] replaced by $+\infty$. We obtain that $Q_{[n/2]+2}(t) \leqslant C(\eta + \frac{\varepsilon^2}{\langle t\rangle^3})$; so we have that

$$M_{n-1}(t) \leqslant C\left(\frac{\varepsilon^2}{\langle t\rangle^3} + (\varepsilon+\eta)\widetilde{Q}_n(t)\right) \quad \text{and} \quad M_{[n/2]+1}(t) \leqslant C\left(\frac{\varepsilon^2}{\langle t\rangle^3} + (\varepsilon+\eta)\eta\right).$$

Hence from (B.5) we obtain that

$$\left\|\sigma_+(t)\tau^a_{13}(t)\right\| \leqslant C\left(\frac{\varepsilon^2}{\langle t\rangle^3} + (\varepsilon+\eta)\widetilde{Q}_n(t)\right). \tag{B.6}$$

Now (B.3) implies that $\|\sigma_+(t)\tau^a_j(t)\| \leqslant C(\eta Q_n(t) + (\eta + \frac{\varepsilon^2}{\langle t\rangle^3})E^{1/2}_{n-1}(t))$ if $j \in \{3, 6, 9, 12\}$, and Proposition 4.1 follows if we also take (B.1), (B.6) into account.

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